



# Sample Paper for the aomart Class

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WITH APPENDIX BY FRODO BAGGINS and AFTERWORD BY BILBO BAGGINS 

## Abstract

This is a test file for `aomart` class based on the `testmath.tex` file from the `amsmath` distribution.

It was changed to test the features of the Annals of Mathematics class.

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Keywords: Hamiltonian paths, Typesetting  
AMS Classification: Primary: 1A5 (matsc2020), 2FD5 (matsc2020); Secondary: FFFF (matsc2020), G25 (matsc2020).  
The class was commissioned by Annals of Mathematics.  
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## 1. Introduction

18

19 This paper demonstrates the use of `aomart` class. It is based on `testmath.tex`  
20 from  $\mathcal{A}\mathcal{M}\mathcal{S}\text{-}\mathcal{L}\mathcal{A}\mathcal{T}\mathcal{E}\mathcal{X}$  distribution. The text is (slightly) reformatted according to  
21 the requirements of the `aomart` style. See also [12], [22], [17], [1], [16], [15],  
22 [24], [23], and [6].

Are these  
 quotations  
 necessary?

23

It is always a pleasure to cite Knuth [9].

2425

## 2. Enumeration of Hamiltonian paths in a graph

26

Let  $\mathbf{A} = (a_{ij})$  be the adjacency matrix of graph  $G$ . The corresponding  
27 Kirchhoff matrix  $\mathbf{K} = (k_{ij})$  is obtained from  $\mathbf{A}$  by replacing in  $-\mathbf{A}$  each  
28 diagonal entry by the degree of its corresponding vertex; i.e., the  $i$ th diagonal  
29 entry is identified with the degree of the  $i$ th vertex. It is well known that  
30

$$\text{(1) } \det \mathbf{K}(i|i) = \text{the number of spanning trees of } G, \quad i = 1, \dots, n$$

32

where  $\mathbf{K}(i|i)$  is the  $i$ th principal submatrix of  $\mathbf{K}$ .

33

`\det\mathbf{K}(i|i)` = the number of spanning trees of  $G$ ,

34

Let  $C_{i(j)}$  be the set of graphs obtained from  $G$  by attaching edge  $(v_i v_j)$   
35 to each spanning tree of  $G$ . Denote by  $C_i = \bigcup_j C_{i(j)}$ . It is obvious that the  
36 collection of Hamiltonian cycles is a subset of  $C_i$ . Note that the cardinality of  
37  $C_i$  is  $k_{ii} \det \mathbf{K}(i|i)$ . Let  $\widehat{X} = \{\hat{x}_1, \dots, \hat{x}_n\}$ .

39

where  $X = \{\hat{x}_1, \dots, \hat{x}_n\}$

40

Define multiplication for the elements of  $\widehat{X}$  by

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$$\text{(2) } \hat{x}_i \hat{x}_j = \hat{x}_j \hat{x}_i, \quad \hat{x}_i^2 = 0, \quad i, j = 1, \dots, n.$$

$\frac{1}{2}$  Let  $\hat{k}_{ij} = k_{ij}\hat{x}_j$  and  $\hat{k}_{ij} = -\sum_{j \neq i} \hat{k}_{ij}$ . Then the number of Hamiltonian cycles  
 $\frac{2}{3}$   $H_c$  is given by the relation [13]

$$\frac{4}{5} \quad (3) \quad \left( \prod_{j=1}^n \hat{x}_j \right) H_c = \frac{1}{2} \hat{k}_{ij} \det \widehat{\mathbf{K}}(i|i), \quad i = 1, \dots, n.$$

$\frac{6}{7}$  The task here is to express (3) in a form free of any  $\hat{x}_i, i = 1, \dots, n$ . The result  
 $\frac{8}{9}$  also leads to the resolution of enumeration of Hamiltonian paths in a graph.

$\frac{10}{11}$  It is well known that the enumeration of Hamiltonian cycles and paths  
 $\frac{12}{13}$  in a complete graph  $K_n$  and in a complete bipartite graph  $K_{n_1 n_2}$  can only  
 $\frac{14}{15}$  be found from *first combinatorial principles* [7]. One wonders if there exists a  
 $\frac{16}{17}$  formula which can be used very efficiently to produce  $K_n$  and  $K_{n_1 n_2}$ . Recently,  
 $\frac{18}{19}$  using Lagrangian methods, Goulden and Jackson have shown that  $H_c$  can be  
 $\frac{20}{21}$  expressed in terms of the determinant and permanent of the adjacency matrix  
 $\frac{22}{23}$  [5]. However, the formula of Goulden and Jackson determines neither  $K_n$  nor  
 $\frac{24}{25}$   $K_{n_1 n_2}$  effectively. In this paper, using an algebraic method, we parametrize  
 $\frac{26}{27}$  the adjacency matrix. The resulting formula also involves the determinant  
 $\frac{28}{29}$  and permanent, but it can easily be applied to  $K_n$  and  $K_{n_1 n_2}$ . In addition,  
 $\frac{30}{31}$  we eliminate the permanent from  $H_c$  and show that  $H_c$  can be represented by  
 $\frac{32}{33}$  a determinantal function of multivariables, each variable with domain  $\{0, 1\}$ .  
 $\frac{34}{35}$  Furthermore, we show that  $H_c$  can be written by number of spanning trees of  
 $\frac{36}{37}$  subgraphs. Finally, we apply the formulas to a complete multigraph  $K_{n_1 \dots n_p}$ .

$\frac{38}{39}$  The conditions  $a_{ij} = a_{ji}, i, j = 1, \dots, n$ , are not required in this paper.  
 $\frac{40}{41}$  All formulas can be extended to a digraph simply by multiplying  $H_c$  by 2.  
 $\frac{42}{43}$  Some other discussion can be found in [4] and [3].

### 3. Main theorem

$\frac{44}{45}$  *Notation.* For  $p, q \in P$  and  $n \in \omega$  we write  $(q, n) \leq (p, n)$  if  $q \leq p$  and  
 $\frac{46}{47}$   $A_{q,n} = A_{p,n}$ .

$\frac{48}{49}$   $\backslash\text{begin}\{\text{notation}\}$  For  $\$p, q \in P\$$  and  $\$n \in \omega\$$

$\frac{50}{51}$   $\dots$

$\frac{52}{53}$   $\backslash\text{end}\{\text{notation}\}$

$\frac{54}{55}$  Let  $\mathbf{B} = (b_{ij})$  be an  $n \times n$  matrix. Let  $\mathbf{n} = \{1, \dots, n\}$ . Using the properties  
 $\frac{56}{57}$  of (2), it is readily seen that

$\frac{58}{59}$  LEMMA 3.1.

$$\frac{60}{61} \quad (4) \quad \prod_{i \in \mathbf{n}} \left( \sum_{j \in \mathbf{n}} b_{ij} \hat{x}_i \right) = \left( \prod_{i \in \mathbf{n}} \hat{x}_i \right) \text{per } \mathbf{B}$$

$\frac{62}{63}$  where  $\text{per } \mathbf{B}$  is the permanent of  $\mathbf{B}$ .

$\frac{1}{2}$  Let  $\widehat{Y} = \{\hat{y}_1, \dots, \hat{y}_n\}$ . Define multiplication for the elements of  $\widehat{Y}$  by

$$\frac{2}{3} \quad (5) \quad \hat{y}_i \hat{y}_j + \hat{y}_j \hat{y}_i = 0, \quad i, j = 1, \dots, n.$$

$\frac{4}{5}$  Then, it follows that

$\frac{6}{7}$  LEMMA 3.2.

$$\frac{7}{8} \quad (6) \quad \prod_{i \in \mathbf{n}} \left( \sum_{j \in \mathbf{n}} b_{ij} \hat{y}_j \right) = \left( \prod_{i \in \mathbf{n}} \hat{y}_i \right) \det \mathbf{B}.$$

$\frac{9}{10}$  Note that all basic properties of determinants are direct consequences of  
 $\frac{11}{12}$  Lemma 3.2. Write

$$\frac{12}{13} \quad (7) \quad \sum_{j \in \mathbf{n}} b_{ij} \hat{y}_j = \sum_{j \in \mathbf{n}} b_{ij}^{(\lambda)} \hat{y}_j + (b_{ii} - \lambda_i) \hat{y}_i$$

$\frac{14}{15}$  where

$$\frac{16}{17} \quad (8) \quad b_{ii}^{(\lambda)} = \lambda_i, \quad b_{ij}^{(\lambda)} = b_{ij}, \quad i \neq j.$$

$\frac{18}{19}$  Let  $\mathbf{B}^{(\lambda)} = (b_{ij}^{(\lambda)})$ . By (6) and (7), it is straightforward to show the following result:

$\frac{20}{21}$  THEOREM 3.3.

$$\frac{22}{23} \quad (9) \quad \det \mathbf{B} = \sum_{l=0}^n \sum_{I_l \subseteq \mathbf{n}} \prod_{i \in I_l} (b_{ii} - \lambda_i) \det \mathbf{B}^{(\lambda)}(I_l | I_l),$$

$\frac{24}{25}$  where  $I_l = \{i_1, \dots, i_l\}$  and  $\mathbf{B}^{(\lambda)}(I_l | I_l)$  is the principal submatrix (obtained from  
 $\frac{26}{27}$   $\mathbf{B}^{(\lambda)}$  by deleting its  $i_1, \dots, i_l$  rows and columns).

$\frac{28}{29}$  Remark 3.1 (convention). Let  $\mathbf{M}$  be an  $n \times n$  matrix. The convention  
 $\frac{30}{31}$   $\mathbf{M}(\mathbf{n} | \mathbf{n}) = 1$  has been used in (9) and hereafter.

$\frac{32}{33}$  Before proceeding with our discussion, we pause to note that Theorem 3.3  
 $\frac{34}{35}$  yields immediately a fundamental formula which can be used to compute the  
 $\frac{36}{37}$  coefficients of a characteristic polynomial [14]:

$\frac{38}{39}$  COROLLARY 3.4. Write  $\det(\mathbf{B} - x\mathbf{I}) = \sum_{l=0}^n (-1)^l b_l x^l$ . Then

$$\frac{40}{41} \quad (10) \quad b_l = \sum_{I_l \subseteq \mathbf{n}} \det \mathbf{B}(I_l | I_l).$$

$\frac{42}{43}$  Let

$$\frac{44}{45} \quad (11) \quad \mathbf{K}(t, t_1, \dots, t_n) = \begin{pmatrix} D_1 t & -a_{12} t_2 & \dots & -a_{1n} t_n \\ -a_{21} t_1 & D_2 t & \dots & -a_{2n} t_n \\ \dots & \dots & \dots & \dots \\ -a_{n1} t_1 & -a_{n2} t_2 & \dots & D_n t \end{pmatrix},$$

$$\begin{pmatrix} D_1 t & -a_{12} t_2 & \dots & -a_{1n} t_n \\ -a_{21} t_1 & D_2 t & \dots & -a_{2n} t_n \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} t_1 & -a_{n2} t_2 & \dots & D_n t \end{pmatrix}$$

where

$$(12) \quad D_i = \sum_{j \in \mathbf{n}} a_{ij} t_j, \quad i = 1, \dots, n.$$

Set

$$D(t_1, \dots, t_n) = \frac{\delta}{\delta t} \det \mathbf{K}(t, t_1, \dots, t_n)|_{t=1}.$$

Then

$$(13) \quad D(t_1, \dots, t_n) = \sum_{i \in \mathbf{n}} D_i \det \mathbf{K}(t = 1, t_1, \dots, t_n; i|i),$$

where  $\mathbf{K}(t = 1, t_1, \dots, t_n; i|i)$  is the  $i$ th principal submatrix of  $\mathbf{K}(t = 1, t_1, \dots, t_n)$ .

[Theorem 3.3](#) leads to

$$(14) \quad \det \mathbf{K}(t_1, t_1, \dots, t_n) = \sum_{I \in \mathbf{n}} (-1)^{|I|} t^{n-|I|} \prod_{i \in I} t_i \prod_{j \in I} (D_j + \lambda_j t_j) \det \mathbf{A}^{(\lambda)}(\overline{I}|\overline{I}).$$

Note that

$$(15) \quad \det \mathbf{K}(t = 1, t_1, \dots, t_n) = \sum_{I \in \mathbf{n}} (-1)^{|I|} \prod_{i \in I} t_i \prod_{j \in I} (D_j + \lambda_j t_j) \det \mathbf{A}^{(\lambda)}(\overline{I}|\overline{I}) = 0.$$

Let  $t_i = \hat{x}_i, i = 1, \dots, n$ . [Lemma 3.1](#) yields

$$(16) \quad \left( \sum_{i \in \mathbf{n}} a_i x_i \right) \det \mathbf{K}(t = 1, x_1, \dots, x_n; l|l) \\ = \left( \prod_{i \in \mathbf{n}} \hat{x}_i \right) \sum_{I \subseteq \mathbf{n} - \{l\}} (-1)^{|I|} \text{per } \mathbf{A}^{(\lambda)}(I|I) \det \mathbf{A}^{(\lambda)}(\overline{I} \cup \{l\} | \overline{I} \cup \{l\}).$$

```

\begin{multline}
\biggl( \sum_{i \in \mathbf{n}} a_i x_i \biggr) \\
\det \mathbf{K}(t=1, x_1, \dots, x_n; l|l) \\
= \biggl( \prod_{i \in \mathbf{n}} \hat{x}_i \biggr) \\
\sum_{I \subseteq \mathbf{n} - \{l\}} \\
(-1)^{|\overline{I}|} \text{per } \mathbf{A}^{(\lambda)}(I|I) \\
\det \mathbf{A}^{(\lambda)} \\
(\overline{I} \cup \{l\} | \overline{I} \cup \{l\}). \\
\label{sum-ali}
\end{multline}

```

By (3), (6), and (7), we have

1 PROPOSITION 3.5.

2  
3 (17) 
$$H_c = \frac{1}{2n} \sum_{l=0}^n (-1)^l D_l,$$

4 where

5  
6 (18) 
$$D_l = \sum_{I_l \subseteq \mathbf{n}} D(t_1, \dots, t_n) 2^{|t_i = \begin{cases} 0, & \text{if } i \in I_l \\ 1, & \text{otherwise} \end{cases}}, \quad i=1, \dots, n.$$

7

8

#### 9 4. Application

10

11 We consider here the applications of [Theorems 5.1 and 5.2](#) on [page 23](#) to  
12 a complete multipartite graph  $K_{n_1 \dots n_p}$ . It can be shown that the number of  
13 spanning trees of  $K_{n_1 \dots n_p}$  may be written

14 (19) 
$$T = n^{p-2} \prod_{i=1}^p (n - n_i)^{n_i - 1}$$

15

16 where

17 (20) 
$$n = n_1 + \dots + n_p.$$

18

19 It follows from [Theorems 5.1 and 5.2](#) that

20  
21 
$$H_c = \frac{1}{2n} \sum_{l=0}^n (-1)^l (n-l)^{p-2} \sum_{l_1 + \dots + l_p = l} \prod_{i=1}^p \binom{n_i}{l_i}$$
  
22  
23 (21) 
$$\cdot [(n-l) - (n_i - l_i)]^{n_i - l_i} \cdot \left[ (n-l)^2 - \sum_{j=1}^p (n_i - l_i)^2 \right].$$

24

25 ...  $\backslash \text{binom}\{n_i\}\{l_i\} \backslash \backslash$

26 and

27

28  
29 
$$H_c = \frac{1}{2} \sum_{l=0}^{n-1} (-1)^l (n-l)^{p-2} \sum_{l_1 + \dots + l_p = l} \prod_{i=1}^p \binom{n_i}{l_i}$$
  
30 (22) 
$$\cdot [(n-l) - (n_i - l_i)]^{n_i - l_i} \left( 1 - \frac{l_p}{n_p} \right) [(n-l) - (n_p - l_p)].$$

31

32 The enumeration of  $H_c$  in a  $K_{n_1 \dots n_p}$  graph can also be carried out by  
33 [Theorem 7.2 or 7.3](#) together with the algebraic method of (2). Some elegant  
34 representations may be obtained. For example,  $H_c$  in a  $K_{n_1 n_2 n_3}$  graph may be  
35 written

36  
37  
38 (23) 
$$H_c = \frac{n_1! n_2! n_3!}{n_1 + n_2 + n_3} \sum_i \left[ \binom{n_1}{i} \binom{n_2}{n_3 - n_1 + i} \binom{n_3}{n_3 - n_2 + i} \right.$$
  
39 
$$\left. + \binom{n_1 - 1}{i} \binom{n_2 - 1}{n_3 - n_1 + i} \binom{n_3 - 1}{n_3 - n_2 + i} \right].$$

40

41

42

## 5. Secret key exchanges

Modern cryptography is fundamentally concerned with the problem of secure private communication. A Secret Key Exchange is a protocol where Alice and Bob, having no secret information in common to start, are able to agree on a common secret key, conversing over a public channel. The notion of a Secret Key Exchange protocol was first introduced in the seminal paper of Diffie and Hellman [2]. [2] presented a concrete implementation of a Secret Key Exchange protocol, dependent on a specific assumption (a variant on the discrete log), specially tailored to yield Secret Key Exchange. Secret Key Exchange is of course trivial if trapdoor permutations exist. However, there is no known implementation based on a weaker general assumption.

The concept of an informationally one-way function was introduced in [8]. We give only an informal definition here:

*Definition 5.1 (one way).* A polynomial time computable function  $f = \{f_k\}$  is informationally one-way if there is no probabilistic polynomial time algorithm which (with probability of the form  $1 - k^{-e}$  for some  $e > 0$ ) returns on input  $y \in \{0, 1\}^k$  a random element of  $f^{-1}(y)$ .

In the non-uniform setting [8] show that these are not weaker than one-way functions:

**THEOREM 5.1** ([8] (non-uniform)). *The existence of informationally one-way functions implies the existence of one-way functions.*

We will stick to the convention introduced above of saying “non-uniform” before the theorem statement when the theorem makes use of non-uniformity. It should be understood that if nothing is said then the result holds for both the uniform and the non-uniform models.

It now follows from Theorem 5.1 that

**THEOREM 5.2** (non-uniform). *Weak SKE implies the existence of a one-way function.*

More recently, the polynomial-time, interior point algorithms for linear programming have been extended to the case of convex quadratic programs [19] and [21], certain linear complementarity problems [11] and [18], and the nonlinear complementarity problem [10]. The connection between these algorithms and the classical Newton method for nonlinear equations is well explained in [11].

## 6. Review

We begin our discussion with the following definition:

1        *Definition 6.1.* A function  $H: \mathfrak{R}^n \rightarrow \mathfrak{R}^n$  is said to be *B-differentiable*  
2 at the point  $z$  if (i)  $H$  is Lipschitz continuous in a neighborhood of  $z$ , and  
3 (ii) there exists a positive homogeneous function  $BH(z): \mathfrak{R}^n \rightarrow \mathfrak{R}^n$ , called the  
4 *B-derivative* of  $H$  at  $z$ , such that

$$\lim_{v \rightarrow 0} \frac{H(z+v) - H(z) - BH(z)v}{\|v\|} = 0.$$

7 The function  $H$  is *B-differentiable in set  $S$*  if it is B-differentiable at every  
8 point in  $S$ . The B-derivative  $BH(z)$  is said to be *strong* if

$$\lim_{(v,v') \rightarrow (0,0)} \frac{H(z+v) - H(z+v') - BH(z)(v-v')}{\|v-v'\|} = 0.$$

12        LEMMA 6.1. *There exists a smooth function  $\psi_0(z)$  defined for  $|z| > 1 - 2a$*   
13 *satisfying the following properties:*

- 14 (i)  $\psi_0(z)$  is bounded above and below by positive constants  $c_1 \leq \psi_0(z) \leq c_2$ .
- 15 (ii) If  $|z| > 1$ , then  $\psi_0(z) = 1$ .
- 16 (iii) For all  $z$  in the domain of  $\psi_0$ ,  $\Delta_0 \ln \psi_0 \geq 0$ .
- 17 (iv) If  $1 - 2a < |z| < 1 - a$ , then  $\Delta_0 \ln \psi_0 \geq c_3 > 0$ .

19        *Proof.* We choose  $\psi_0(z)$  to be a radial function depending only on  $r = |z|$ .  
20 Let  $h(r) \geq 0$  be a suitable smooth function satisfying  $h(r) \geq c_3$  for  $1 - 2a <$   
21  $|z| < 1 - a$ , and  $h(r) = 0$  for  $|z| > 1 - \frac{a}{2}$ . The radial Laplacian

$$\Delta_0 \ln \psi_0(r) = \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) \ln \psi_0(r)$$

24 has smooth coefficients for  $r > 1 - 2a$ . Therefore, we may apply the existence  
25 and uniqueness theory for ordinary differential equations. Simply let  $\ln \psi_0(r)$   
26 be the solution of the differential equation

$$\left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) \ln \psi_0(r) = h(r)$$

28 with initial conditions given by  $\ln \psi_0(1) = 0$  and  $\ln \psi_0'(1) = 0$ .

31        Next, let  $D_\nu$  be a finite collection of pairwise disjoint disks, all of which  
32 are contained in the unit disk centered at the origin in  $C$ . We assume that  
33  $D_\nu = \{z \mid |z - z_\nu| < \delta\}$ . Suppose that  $D_\nu(a)$  denotes the smaller concentric  
34 disk  $D_\nu(a) = \{z \mid |z - z_\nu| \leq (1 - 2a)\delta\}$ . We define a smooth weight function  
35  $\Phi_0(z)$  for  $z \in C - \bigcup_\nu D_\nu(a)$  by setting  $\Phi_0(z) = 1$  when  $z \notin \bigcup_\nu D_\nu$  and  
36  $\Phi_0(z) = \psi_0((z - z_\nu)/\delta)$  when  $z$  is an element of  $D_\nu$ . It follows from Lemma 6.1  
37 that  $\Phi_0$  satisfies the properties:

- 38 (i)  $\Phi_0(z)$  is bounded above and below by positive constants  $c_1 \leq \Phi_0(z) \leq c_2$ .
- 39 (ii)  $\Delta_0 \ln \Phi_0 \geq 0$  for all  $z \in C - \bigcup_\nu D_\nu(a)$ , the domain where the function  $\Phi_0$   
40 is defined.
- 41 (iii)  $\Delta_0 \ln \Phi_0 \geq c_3 \delta^{-2}$  when  $(1 - 2a)\delta < |z - z_\nu| < (1 - a)\delta$ .



$\frac{1}{2}$  Let  $A_\nu$  denote the annulus  $A_\nu = \{(1 - 2a)\delta < |z - z_\nu| < (1 - a)\delta\}$ , and  
 $\frac{2}{3}$  set  $A = \bigcup_\nu A_\nu$ . The [properties \(2\)](#) and [\(3\)](#) of  $\Phi_0$  may be summarized as  
 $\frac{3}{4}$   $\Delta_0 \ln \Phi_0 \geq c_3 \delta^{-2} \chi_A$ , where  $\chi_A$  is the characteristic function of  $A$ .  $\square$

$\frac{5}{6}$  Suppose that  $\alpha$  is a nonnegative real constant. We apply [Proposition 3.5](#)  
 $\frac{6}{7}$  with  $\Phi(z) = \Phi_0(z)e^{\alpha|z|^2}$ . If  $u \in C_0^\infty(R^2 - \bigcup_\nu D_\nu(a))$ , assume that  $\mathcal{D}$  is a  
 $\frac{7}{8}$  bounded domain containing the support of  $u$  and  $A \subset \mathcal{D} \subset R^2 - \bigcup_\nu D_\nu(a)$ . A  
 $\frac{8}{9}$  calculation gives

$$\frac{9}{10} \int_{\mathcal{D}} |\bar{\partial}u|^2 \Phi_0(z)e^{\alpha|z|^2} \geq c_4 \alpha \int_{\mathcal{D}} |u|^2 \Phi_0 e^{\alpha|z|^2} + c_5 \delta^{-2} \int_A |u|^2 \Phi_0 e^{\alpha|z|^2}.$$

$\frac{11}{12}$  The boundedness, [property \(1\)](#) of  $\Phi_0$ , then yields

$$\frac{13}{14} \int_{\mathcal{D}} |\bar{\partial}u|^2 e^{\alpha|z|^2} \geq c_6 \alpha \int_{\mathcal{D}} |u|^2 e^{\alpha|z|^2} + c_7 \delta^{-2} \int_A |u|^2 e^{\alpha|z|^2}.$$

$\frac{15}{16}$  Let  $B(X)$  be the set of blocks of  $\Lambda_X$  and let  $b(X) = |B(X)|$ . If  $\phi \in Q_X$   
 $\frac{17}{18}$  then  $\phi$  is constant on the blocks of  $\Lambda_X$ .

$$\frac{18}{19} (24) \quad P_X = \{\phi \in M \mid \Lambda_\phi = \Lambda_X\}, \quad Q_X = \{\phi \in M \mid \Lambda_\phi \geq \Lambda_X\}.$$

$\frac{20}{21}$  If  $\Lambda_\phi \geq \Lambda_X$  then  $\Lambda_\phi = \Lambda_Y$  for some  $Y \geq X$  so that

$$\frac{22}{23} Q_X = \bigcup_{Y \geq X} P_Y.$$

$\frac{24}{25}$  Thus by Möbius inversion

$$\frac{26}{27} |P_Y| = \sum_{X \geq Y} \mu(Y, X) |Q_X|.$$

$\frac{28}{29}$  Thus there is a bijection from  $Q_X$  to  $W^{B(X)}$ . In particular  $|Q_X| = w^{b(X)}$ .

$\frac{30}{31}$  Next note that  $b(X) = \dim X$ . We see this by choosing a basis for  $X$   
 $\frac{31}{32}$  consisting of vectors  $v^k$  defined by

$$\frac{32}{33} v_i^k = \begin{cases} 1 & \text{if } i \in \Lambda_k, \\ 0 & \text{otherwise.} \end{cases}$$

$\frac{34}{35}$   $\backslash [v^{\{k\}}_{\{i\}} =$   
 $\frac{36}{37}$   $\backslash \begin{cases} 1 & \text{if } i \in \Lambda_{\{k\}}, \\ 0 & \text{otherwise.} \end{cases} \backslash \backslash$   
 $\frac{38}{39}$   $\backslash ]$

$\frac{39}{40}$  LEMMA 6.2. *Let  $\mathcal{A}$  be an arrangement. Then*

$$\frac{41}{42} \chi(\mathcal{A}, t) = \sum_{\mathcal{B} \subseteq \mathcal{A}} (-1)^{|\mathcal{B}|} t^{\dim T(\mathcal{B})}.$$

1 In order to compute  $R''$  recall the definition of  $S(X, Y)$  from Lemma 3.1.  
2 Since  $H \in \mathcal{B}$ ,  $\mathcal{A}_H \subseteq \mathcal{B}$ . Thus if  $T(\mathcal{B}) = Y$  then  $\mathcal{B} \in S(H, Y)$ . Let  $L'' = L(\mathcal{A}'')$ .  
3 Then

$$\begin{aligned}
 \frac{4}{R''} &= \sum_{H \in \mathcal{B} \subseteq \mathcal{A}} (-1)^{|\mathcal{B}|} t^{\dim T(\mathcal{B})} \\
 \frac{5}{} &= \sum_{Y \in L''} \sum_{\mathcal{B} \in S(H, Y)} (-1)^{|\mathcal{B}|} t^{\dim Y} \\
 \frac{6}{} &= - \sum_{Y \in L''} \sum_{\mathcal{B} \in S(H, Y)} (-1)^{|\mathcal{B} - \mathcal{A}_H|} t^{\dim Y} \\
 \frac{7}{(25)} &= - \sum_{Y \in L''} \mu(H, Y) t^{\dim Y} \\
 \frac{8}{} &= -\chi(\mathcal{A}'', t).
 \end{aligned}$$

9 COROLLARY 6.3. Let  $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$  be a triple of arrangements. Then

$$\pi(\mathcal{A}, t) = \pi(\mathcal{A}', t) + t\pi(\mathcal{A}'', t).$$

10 *Definition 6.2.* Let  $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$  be a triple with respect to the hyperplane  
11  $H \in \mathcal{A}$ . Call  $H$  a *separator* if  $T(\mathcal{A}) \notin L(\mathcal{A}')$ .  
12

13 COROLLARY 6.4. Let  $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$  be a triple with respect to  $H \in \mathcal{A}$ .

14 (i) If  $H$  is a separator then

$$\mu(\mathcal{A}) = -\mu(\mathcal{A}'')$$

15 and hence

$$|\mu(\mathcal{A})| = |\mu(\mathcal{A}'')|.$$

16 (ii) If  $H$  is not a separator then

$$\mu(\mathcal{A}) = \mu(\mathcal{A}') - \mu(\mathcal{A}'')$$

17 and

$$|\mu(\mathcal{A})| = |\mu(\mathcal{A}')| + |\mu(\mathcal{A}'')|.$$

18 *Proof.* It follows from Theorem 5.1 that  $\pi(\mathcal{A}, t)$  has leading term

$$(-1)^{r(\mathcal{A})} \mu(\mathcal{A}) t^{r(\mathcal{A})}.$$

19 The conclusion follows by comparing coefficients of the leading terms on both  
20 sides of the equation in Corollary 6.3. If  $H$  is a separator then  $r(\mathcal{A}') < r(\mathcal{A})$   
21 and there is no contribution from  $\pi(\mathcal{A}', t)$ .  $\square$

22 The Poincaré polynomial of an arrangement will appear repeatedly in  
23 these notes. It will be shown to equal the Poincaré polynomial of the graded  
24 algebras which we are going to associate with  $\mathcal{A}$ . It is also the Poincaré poly-  
25 nomial of the complement  $M(\mathcal{A})$  for a complex arrangement. Here we prove  
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Figure 1.  $Q(\mathcal{A}_1) = xyz(x-z)(x+z)(y-z)(y+z)$

Figure 2.  $Q(\mathcal{A}_2) = xyz(x+y+z)(x+y-z)(x-y+z)(x-y-z)$

that the Poincaré polynomial is the chamber counting function for a real arrangement. The complement  $M(\mathcal{A})$  is a disjoint union of chambers

$$M(\mathcal{A}) = \bigcup_{C \in \text{Cham}(\mathcal{A})} C.$$

The number of chambers is determined by the Poincaré polynomial as follows.

**THEOREM 6.5.** *Let  $\mathcal{A}_{\mathbf{R}}$  be a real arrangement. Then*

$$|\text{Cham}(\mathcal{A}_{\mathbf{R}})| = \pi(\mathcal{A}_{\mathbf{R}}, 1).$$

*Proof.* We check the properties required in [Corollary 6.4](#): (i) follows from  $\pi(\Phi_l, t) = 1$ , and (ii) is a consequence of [Corollary 3.4](#).  $\square$

**THEOREM 6.6.** *Let  $\phi$  be a protocol for a random pair  $(X, Y)$ . If one of  $\sigma_\phi(x', y)$  and  $\sigma_\phi(x, y')$  is a prefix of the other and  $(x, y) \in S_{X, Y}$ , then*

$$\langle \sigma_j(x', y) \rangle_{j=1}^\infty = \langle \sigma_j(x, y) \rangle_{j=1}^\infty = \langle \sigma_j(x, y') \rangle_{j=1}^\infty.$$

$\frac{1}{2}$  *Proof.* We show by induction on  $i$  that

$$\frac{3}{4} \quad \langle \sigma_j(x', y) \rangle_{j=1}^i = \langle \sigma_j(x, y) \rangle_{j=1}^i = \langle \sigma_j(x, y') \rangle_{j=1}^i.$$

$\frac{4}{5}$  The induction hypothesis holds vacuously for  $i = 0$ . Assume it holds for  
 $\frac{5}{6}$   $i - 1$ , in particular  $[\sigma_j(x', y)]_{j=1}^{i-1} = [\sigma_j(x, y')]_{j=1}^{i-1}$ . Then one of  $[\sigma_j(x', y)]_{j=i}^{\infty}$   
 $\frac{6}{7}$  and  $[\sigma_j(x, y')]_{j=i}^{\infty}$  is a prefix of the other which implies that one of  $\sigma_i(x', y)$   
 $\frac{7}{8}$  and  $\sigma_i(x, y')$  is a prefix of the other. If the  $i$ th message is transmitted by  
 $\frac{8}{9}$   $P_X$  then, by the separate-transmissions property and the induction hypothe-  
 $\frac{9}{10}$  sis,  $\sigma_i(x, y) = \sigma_i(x, y')$ , hence one of  $\sigma_i(x, y)$  and  $\sigma_i(x', y)$  is a prefix of the  
 $\frac{10}{11}$  other. By the implicit-termination property, neither  $\sigma_i(x, y)$  nor  $\sigma_i(x', y)$  can  
 $\frac{11}{12}$  be a proper prefix of the other, hence they must be the same and  $\sigma_i(x', y) =$   
 $\frac{12}{13}$   $\sigma_i(x, y) = \sigma_i(x, y')$ . If the  $i$ th message is transmitted by  $P_Y$  then, symmet-  
 $\frac{13}{14}$  rically,  $\sigma_i(x, y) = \sigma_i(x', y)$  by the induction hypothesis and the separate-  
 $\frac{14}{15}$  transmissions property, and, then,  $\sigma_i(x, y) = \sigma_i(x, y')$  by the implicit-termination  
 $\frac{15}{16}$  property, proving the induction step.  $\square$

$\frac{16}{17}$  If  $\phi$  is a protocol for  $(X, Y)$ , and  $(x, y)$ ,  $(x', y)$  are distinct inputs in  $S_{X,Y}$ ,  
 $\frac{17}{18}$  then, by the correct-decision property,  $\langle \sigma_j(x, y) \rangle_{j=1}^{\infty} \neq \langle \sigma_j(x', y) \rangle_{j=1}^{\infty}$ .

$\frac{18}{19}$  Equation (25) defined  $P_Y$ 's ambiguity set  $S_{X|Y}(y)$  to be the set of possible  
 $\frac{19}{20}$   $X$  values when  $Y = y$ . The last corollary implies that for all  $y \in S_Y$ , the  
 $\frac{20}{21}$  multiset<sup>1</sup> of codewords  $\{\sigma_\phi(x, y) : x \in S_{X|Y}(y)\}$  is prefix free.

$\frac{21}{22}$

$\frac{22}{23}$

## 7. One-way complexity

$\frac{23}{24}$   $\hat{C}_1(X|Y)$ , the one-way complexity of a random pair  $(X, Y)$ , is the number  
 $\frac{24}{25}$  of bits  $P_X$  must transmit in the worst case when  $P_Y$  is not permitted to transmit  
 $\frac{25}{26}$  any feedback messages. Starting with  $S_{X,Y}$ , the support set of  $(X, Y)$ , we define  
 $\frac{26}{27}$   $G(X|Y)$ , the *characteristic hypergraph* of  $(X, Y)$ , and show that  
 $\frac{27}{28}$

$$\frac{28}{29} \quad \hat{C}_1(X|Y) = \lceil \log \chi(G(X|Y)) \rceil .$$

$\frac{29}{30}$  Let  $(X, Y)$  be a random pair. For each  $y$  in  $S_Y$ , the support set of  $Y$ ,  
 $\frac{30}{31}$  equation (25) defined  $S_{X|Y}(y)$  to be the set of possible  $x$  values when  $Y = y$ .  
 $\frac{31}{32}$  The *characteristic hypergraph*  $G(X|Y)$  of  $(X, Y)$  has  $S_X$  as its vertex set and  
 $\frac{32}{33}$  the hyperedge  $S_{X|Y}(y)$  for each  $y \in S_Y$ .

$\frac{33}{34}$  We can now prove a continuity theorem.

$\frac{34}{35}$  THEOREM 7.1. *Let  $\Omega \subset \mathbf{R}^n$  be an open set, let  $u \in BV(\Omega; \mathbf{R}^m)$ , and let*

$$\frac{35}{36} \quad (26) \quad T_x^u = \left\{ y \in \mathbf{R}^m : y = \tilde{u}(x) + \left\langle \frac{Du}{|Du|}(x), z \right\rangle \text{ for some } z \in \mathbf{R}^n \right\}$$

$\frac{36}{37}$

$\frac{37}{38}$   $\frac{38}{39}$   $\frac{39}{40}$   $\frac{40}{41}$   $\frac{41}{42}$  <sup>1</sup>A multiset allows multiplicity of elements. Hence,  $\{0, 01, 01\}$  is prefix free as a set, but  
not as a multiset.

$\frac{1}{2}$  for every  $x \in \Omega \setminus S_u$ . Let  $f: \mathbf{R}^m \rightarrow \mathbf{R}^k$  be a Lipschitz continuous function such  
 $\frac{2}{2}$  that  $f(0) = 0$ , and let  $v = f(u): \Omega \rightarrow \mathbf{R}^k$ . Then  $v \in BV(\Omega; \mathbf{R}^k)$  and

$$\frac{3}{4} \quad (27) \quad Jv = (f(u^+) - f(u^-)) \otimes \nu_u \cdot \mathcal{H}_{n-1}|_{S_u}.$$

$\frac{5}{6}$  In addition, for  $|\widetilde{Du}|$ -almost every  $x \in \Omega$  the restriction of the function  $f$  to  
 $\frac{7}{7}$   $T_x^u$  is differentiable at  $\tilde{u}(x)$  and

$$\frac{8}{9} \quad (28) \quad \widetilde{Dv} = \nabla(f|_{T_x^u})(\tilde{u}) \frac{\widetilde{Du}}{|\widetilde{Du}|} \cdot |\widetilde{Du}|.$$

$\frac{11}{12}$  Before proving the theorem, we state without proof three elementary re-  
 $\frac{12}{12}$  marks which will be useful in the sequel.

$\frac{13}{14}$  *Remark 7.1.* Let  $\omega: ]0, +\infty[ \rightarrow ]0, +\infty[$  be a continuous function such  
 $\frac{15}{15}$  that  $\omega(t) \rightarrow 0$  as  $t \rightarrow 0$ . Then

$$\frac{16}{17} \quad \lim_{h \rightarrow 0^+} g(\omega(h)) = L \Leftrightarrow \lim_{h \rightarrow 0^+} g(h) = L$$

$\frac{18}{19}$  for any function  $g: ]0, +\infty[ \rightarrow \mathbf{R}$ .

$\frac{20}{21}$  *Remark 7.2.* Let  $g: \mathbf{R}^n \rightarrow \mathbf{R}$  be a Lipschitz continuous function and as-  
 $\frac{21}{21}$  sume that

$$\frac{22}{23} \quad L(z) = \lim_{h \rightarrow 0^+} \frac{g(hz) - g(0)}{h}$$

$\frac{24}{25}$  exists for every  $z \in \mathbf{Q}^n$  and that  $L$  is a linear function of  $z$ . Then  $g$  is differ-  
 $\frac{25}{25}$  entiable at 0.

$\frac{26}{27}$  *Remark 7.3.* Let  $A: \mathbf{R}^n \rightarrow \mathbf{R}^m$  be a linear function, and let  $f: \mathbf{R}^m \rightarrow \mathbf{R}$   
 $\frac{27}{27}$  be a function. Then the restriction of  $f$  to the range of  $A$  is differentiable at 0  
 $\frac{28}{28}$  if and only if  $f(A): \mathbf{R}^n \rightarrow \mathbf{R}$  is differentiable at 0 and

$$\frac{29}{30} \quad \nabla(f|_{\text{Im}(A)})(0)A = \nabla(f(A))(0).$$

$\frac{31}{32}$  *Proof.* We begin by showing that  $v \in BV(\Omega; \mathbf{R}^k)$  and

$$\frac{33}{34} \quad (29) \quad |Dv|(B) \leq K |Du|(B) \quad \forall B \in \mathbf{B}(\Omega),$$

$\frac{35}{36}$  where  $K > 0$  is the Lipschitz constant of  $f$ . By (13) and by the approxima-  
 $\frac{36}{37}$  tion result quoted in §3, it is possible to find a sequence  $(u_h) \subset C^1(\Omega; \mathbf{R}^m)$   
 $\frac{37}{37}$  converging to  $u$  in  $L^1(\Omega; \mathbf{R}^m)$  and such that

$$\frac{38}{39} \quad \lim_{h \rightarrow +\infty} \int_{\Omega} |\nabla u_h| \, dx = |Du|(\Omega).$$

$\frac{40}{41}$  The functions  $v_h = f(u_h)$  are locally Lipschitz continuous in  $\Omega$ , and the defini-  
 $\frac{41}{42}$  tion of differential implies that  $|\nabla v_h| \leq K |\nabla u_h|$  almost everywhere in  $\Omega$ . The

1 lower semicontinuity of the total variation and (13) yield

$$\begin{aligned}
2 \\
3 \quad |Dv|(\Omega) &\leq \liminf_{h \rightarrow +\infty} |Dv_h|(\Omega) = \liminf_{h \rightarrow +\infty} \int_{\Omega} |\nabla v_h| \, dx \\
4 \quad (30) \\
5 &\leq K \liminf_{h \rightarrow +\infty} \int_{\Omega} |\nabla u_h| \, dx = K |Du|(\Omega). \\
6
\end{aligned}$$

7 Since  $f(0) = 0$ , we have also

$$\begin{aligned}
8 \\
9 \quad \int_{\Omega} |v| \, dx &\leq K \int_{\Omega} |u| \, dx; \\
10
\end{aligned}$$

11 therefore  $u \in BV(\Omega; \mathbf{R}^k)$ . Repeating the same argument for every open set  
12  $A \subset \Omega$ , we get (29) for every  $B \in \mathbf{B}(\Omega)$ , because  $|Dv|$ ,  $|Du|$  are Radon mea-  
13 sures. To prove Lemma 6.1, first we observe that

$$\begin{aligned}
14 \\
15 \quad (31) \quad S_v \subset S_u, \quad \tilde{v}(x) = f(\tilde{u}(x)) \quad \forall x \in \Omega \setminus S_u.
\end{aligned}$$

16 In fact, for every  $\varepsilon > 0$  we have

$$\begin{aligned}
17 \\
18 \quad \{y \in B_{\rho}(x) : |v(y) - f(\tilde{u}(x))| > \varepsilon\} \subset \{y \in B_{\rho}(x) : |u(y) - \tilde{u}(x)| > \varepsilon/K\}, \\
19
\end{aligned}$$

20 hence

$$\begin{aligned}
21 \quad \lim_{\rho \rightarrow 0^+} \frac{|\{y \in B_{\rho}(x) : |v(y) - f(\tilde{u}(x))| > \varepsilon\}|}{\rho^n} = 0 \\
22
\end{aligned}$$

23 whenever  $x \in \Omega \setminus S_u$ . By a similar argument, if  $x \in S_u$  is a point such that  
24 there exists a triplet  $(u^+, u^-, \nu_u)$  satisfying (14), (15), then

$$\begin{aligned}
25 \quad (v^+(x) - v^-(x)) \otimes \nu_v &= (f(u^+(x)) - f(u^-(x))) \otimes \nu_u \quad \text{if } x \in S_v \\
26
\end{aligned}$$

27 and  $f(u^-(x)) = f(u^+(x))$  if  $x \in S_u \setminus S_v$ . Hence, by (1.8) we get

$$\begin{aligned}
28 \quad Jv(B) &= \int_{B \cap S_v} (v^+ - v^-) \otimes \nu_v \, d\mathcal{H}_{n-1} = \int_{B \cap S_v} (f(u^+) - f(u^-)) \otimes \nu_u \, d\mathcal{H}_{n-1} \\
29 \\
30 &= \int_{B \cap S_u} (f(u^+) - f(u^-)) \otimes \nu_u \, d\mathcal{H}_{n-1} \\
31 \\
32
\end{aligned}$$

33 and Lemma 6.1 is proved. □

34 To prove (31), it is not restrictive to assume that  $k = 1$ . Moreover, to  
35 simplify our notation, from now on we shall assume that  $\Omega = \mathbf{R}^n$ . The proof  
36 of (31) is divided into two steps. In the first step we prove the statement in  
37 the one-dimensional case ( $n = 1$ ), using Theorem 5.2. In the second step we  
38 achieve the general result using Theorem 7.1.  
39

40 *Step 1.* Assume that  $n = 1$ . Since  $S_u$  is at most countable, (7) yields  
41 that  $|\tilde{D}v|(S_u \setminus S_v) = 0$ , so that (19) and (21) imply that  $Dv = \tilde{D}v + Jv$  is the  
42

$\frac{1}{2}$  Radon-Nikodým decomposition of  $Dv$  in absolutely continuous and singular  
 $\frac{2}{3}$  part with respect to  $|\widetilde{D}u|$ . By Theorem 5.2, we have

$$\frac{4}{5} \quad \frac{\widetilde{D}v}{|\widetilde{D}u|}(t) = \lim_{s \rightarrow t^+} \frac{Dv([t, s])}{|\widetilde{D}u|([t, s])}, \quad \frac{\widetilde{D}u}{|\widetilde{D}u|}(t) = \lim_{s \rightarrow t^+} \frac{Du([t, s])}{|\widetilde{D}u|([t, s])}$$

$\frac{7}{8}$   $|\widetilde{D}u|$ -almost everywhere in  $\mathbf{R}$ . It is well known (see, for instance, [20, 2.5.16])  
 $\frac{9}{10}$  that every one-dimensional function of bounded variation  $w$  has a unique left  
 $\frac{11}{11}$  continuous representative, i.e., a function  $\hat{w}$  such that  $\hat{w} = w$  almost every-  
 $\frac{12}{12}$  where and  $\lim_{s \rightarrow t^-} \hat{w}(s) = \hat{w}(t)$  for every  $t \in \mathbf{R}$ . These conditions imply

$$\frac{13}{13} \quad (32) \quad \hat{u}(t) = Du(-\infty, t], \quad \hat{v}(t) = Dv(-\infty, t] \quad \forall t \in \mathbf{R}$$

$\frac{14}{14}$  and

$$\frac{15}{16} \quad (33) \quad \hat{v}(t) = f(\hat{u}(t)) \quad \forall t \in \mathbf{R}.$$

$\frac{17}{18}$  Let  $t \in \mathbf{R}$  be such that  $|\widetilde{D}u|([t, s]) > 0$  for every  $s > t$  and assume that the  
 $\frac{19}{19}$  limits in (22) exist. By (23) and (24) we get

$$\begin{aligned} \frac{20}{21} \quad \frac{\hat{v}(s) - \hat{v}(t)}{|\widetilde{D}u|([t, s])} &= \frac{f(\hat{u}(s)) - f(\hat{u}(t))}{|\widetilde{D}u|([t, s])} \\ \frac{22}{23} &= \frac{f(\hat{u}(s)) - f(\hat{u}(t) + \frac{\widetilde{D}u}{|\widetilde{D}u|}(t) |\widetilde{D}u|([t, s]))}{|\widetilde{D}u|([t, s])} \\ \frac{24}{25} &+ \frac{f(\hat{u}(t) + \frac{\widetilde{D}u}{|\widetilde{D}u|}(t) |\widetilde{D}u|([t, s])) - f(\hat{u}(t))}{|\widetilde{D}u|([t, s])} \end{aligned}$$

$\frac{32}{32}$  for every  $s > t$ . Using the Lipschitz condition on  $f$  we find

$$\begin{aligned} \frac{33}{34} \quad &\left| \frac{\hat{v}(s) - \hat{v}(t)}{|\widetilde{D}u|([t, s])} - \frac{f(\hat{u}(t) + \frac{\widetilde{D}u}{|\widetilde{D}u|}(t) |\widetilde{D}u|([t, s])) - f(\hat{u}(t))}{|\widetilde{D}u|([t, s])} \right| \\ \frac{35}{36} & \\ \frac{37}{38} & \\ \frac{39}{40} & \\ \frac{41}{42} & \leq K \left| \frac{\hat{u}(s) - \hat{u}(t)}{|\widetilde{D}u|([t, s])} - \frac{\widetilde{D}u}{|\widetilde{D}u|}(t) \right|. \end{aligned}$$

$\frac{1}{2}$  By (29), the function  $s \rightarrow \left| \widetilde{D}u \right| ([t, s])$  is continuous and converges to 0 as  $s \downarrow t$ .  
 $\frac{2}{3}$  Therefore Remark 7.1 and the previous inequality imply

$$\frac{4}{5} \quad \frac{\widetilde{D}v}{\left| \widetilde{D}u \right|}(t) = \lim_{h \rightarrow 0^+} \frac{f(\hat{u}(t) + h \frac{\widetilde{D}u}{\left| \widetilde{D}u \right|}(t)) - f(\hat{u}(t))}{h} \quad \left| \widetilde{D}u \right| \text{-a.e. in } \mathbf{R}.$$

$\frac{8}{9}$  By (22),  $\hat{u}(x) = \tilde{u}(x)$  for every  $x \in \mathbf{R} \setminus S_u$ ; moreover, applying the same argument to the functions  $u'(t) = u(-t)$ ,  $v'(t) = f(u'(t)) = v(-t)$ , we get

$$\frac{11}{12} \quad \frac{\widetilde{D}v}{\left| \widetilde{D}u \right|}(t) = \lim_{h \rightarrow 0} \frac{f(\tilde{u}(t) + h \frac{\widetilde{D}u}{\left| \widetilde{D}u \right|}(t)) - f(\tilde{u}(t))}{h} \quad \left| \widetilde{D}u \right| \text{-a.e. in } \mathbf{R}$$

$\frac{15}{16}$  and our statement is proved.

$\frac{17}{18}$  *Step 2.* Let us consider now the general case  $n > 1$ . Let  $\nu \in \mathbf{R}^n$  be such  
 $\frac{19}{20}$  that  $|\nu| = 1$ , and let  $\pi_\nu = \{y \in \mathbf{R}^n : \langle y, \nu \rangle = 0\}$ . In the following, we shall  
 $\frac{21}{22}$  identify  $\mathbf{R}^n$  with  $\pi_\nu \times \mathbf{R}$ , and we shall denote by  $y$  the variable ranging in  $\pi_\nu$   
 $\frac{23}{24}$  and by  $t$  the variable ranging in  $\mathbf{R}$ . By the just proven one-dimensional result,  
 $\frac{25}{26}$  and by Theorem 3.3, we get

$$\frac{27}{28} \quad \lim_{h \rightarrow 0} \frac{f(\tilde{u}(y + t\nu) + h \frac{\widetilde{D}u_y}{\left| \widetilde{D}u_y \right|}(t)) - f(\tilde{u}(y + t\nu))}{h} = \frac{\widetilde{D}v_y}{\left| \widetilde{D}u_y \right|}(t) \quad \left| \widetilde{D}u_y \right| \text{-a.e. in } \mathbf{R}$$

$\frac{29}{30}$  for  $\mathcal{H}_{n-1}$ -almost every  $y \in \pi_\nu$ . We claim that

$$\frac{31}{32} \quad (34) \quad \frac{\langle \widetilde{D}u, \nu \rangle}{\left| \langle \widetilde{D}u, \nu \rangle \right|}(y + t\nu) = \frac{\widetilde{D}u_y}{\left| \widetilde{D}u_y \right|}(t) \quad \left| \widetilde{D}u_y \right| \text{-a.e. in } \mathbf{R}$$

$\frac{33}{34}$  for  $\mathcal{H}_{n-1}$ -almost every  $y \in \pi_\nu$ . In fact, by (16) and (18) we get

$$\begin{aligned} \frac{35}{36} \quad & \int_{\pi_\nu} \frac{\widetilde{D}u_y}{\left| \widetilde{D}u_y \right|} \cdot \left| \widetilde{D}u_y \right| d\mathcal{H}_{n-1}(y) = \int_{\pi_\nu} \widetilde{D}u_y d\mathcal{H}_{n-1}(y) \\ \frac{37}{38} \quad & = \langle \widetilde{D}u, \nu \rangle = \frac{\langle \widetilde{D}u, \nu \rangle}{\left| \langle \widetilde{D}u, \nu \rangle \right|} \cdot \left| \langle \widetilde{D}u, \nu \rangle \right| = \int_{\pi_\nu} \frac{\langle \widetilde{D}u, \nu \rangle}{\left| \langle \widetilde{D}u, \nu \rangle \right|}(y + \cdot \nu) \cdot \left| \widetilde{D}u_y \right| d\mathcal{H}_{n-1}(y) \end{aligned}$$

$\frac{39}{40}$  and (24) follows from (13). By the same argument it is possible to prove that

$$\frac{41}{42} \quad (35) \quad \frac{\langle \widetilde{D}v, \nu \rangle}{\left| \langle \widetilde{D}v, \nu \rangle \right|}(y + t\nu) = \frac{\widetilde{D}v_y}{\left| \widetilde{D}u_y \right|}(t) \quad \left| \widetilde{D}u_y \right| \text{-a.e. in } \mathbf{R}$$



$\frac{1}{2}$  for  $\mathcal{H}_{n-1}$ -almost every  $y \in \pi_\nu$ . By (24) and (25) we get

$$\frac{3}{4} \quad f(\tilde{u}(y + t\nu) + h \frac{\langle \tilde{D}u, \nu \rangle}{|\langle \tilde{D}u, \nu \rangle|}(y + t\nu)) - f(\tilde{u}(y + t\nu))$$

$$\frac{5}{6} \quad \lim_{h \rightarrow 0} \frac{\quad}{h} = \frac{\langle \tilde{D}v, \nu \rangle}{|\langle \tilde{D}u, \nu \rangle|}(y + t\nu)$$

$\frac{7}{8}$  for  $\mathcal{H}_{n-1}$ -almost every  $y \in \pi_\nu$ , and using again (14), (15) we get

$$\frac{9}{10} \quad f(\tilde{u}(x) + h \frac{\langle \tilde{D}u, \nu \rangle}{|\langle \tilde{D}u, \nu \rangle|}(x)) - f(\tilde{u}(x))$$

$$\frac{11}{12} \quad \lim_{h \rightarrow 0} \frac{\quad}{h} = \frac{\langle \tilde{D}v, \nu \rangle}{|\langle \tilde{D}u, \nu \rangle|}(x)$$

$\frac{13}{14}$   $|\langle \tilde{D}u, \nu \rangle|$ -a.e. in  $\mathbf{R}^n$ .

$\frac{15}{16}$  Since the function  $|\langle \tilde{D}u, \nu \rangle| / |\tilde{D}u|$  is strictly positive  $|\langle \tilde{D}u, \nu \rangle|$ -almost ev-  
 $\frac{17}{18}$  erywhere, we obtain also

$$\frac{19}{20} \quad f(\tilde{u}(x) + h \frac{|\langle \tilde{D}u, \nu \rangle|}{|\tilde{D}u|}(x) \frac{\langle \tilde{D}u, \nu \rangle}{|\langle \tilde{D}u, \nu \rangle|}(x)) - f(\tilde{u}(x))$$

$$\frac{21}{22} \quad \lim_{h \rightarrow 0} \frac{\quad}{h}$$

$$\frac{23}{24} \quad = \frac{|\langle \tilde{D}u, \nu \rangle|}{|\tilde{D}u|}(x) \frac{\langle \tilde{D}v, \nu \rangle}{|\langle \tilde{D}u, \nu \rangle|}(x)$$

$\frac{25}{26}$   $|\langle \tilde{D}u, \nu \rangle|$ -almost everywhere in  $\mathbf{R}^n$ .

$\frac{27}{28}$  Finally, since

$$\frac{29}{30} \quad \frac{|\langle \tilde{D}u, \nu \rangle|}{|\tilde{D}u|} \frac{\langle \tilde{D}u, \nu \rangle}{|\langle \tilde{D}u, \nu \rangle|} = \frac{\langle \tilde{D}u, \nu \rangle}{|\tilde{D}u|} = \left\langle \frac{\tilde{D}u}{|\tilde{D}u|}, \nu \right\rangle \quad |\tilde{D}u| \text{-a.e. in } \mathbf{R}^n$$

$$\frac{31}{32} \quad \frac{|\langle \tilde{D}u, \nu \rangle|}{|\tilde{D}u|} \frac{\langle \tilde{D}v, \nu \rangle}{|\langle \tilde{D}u, \nu \rangle|} = \frac{\langle \tilde{D}v, \nu \rangle}{|\tilde{D}u|} = \left\langle \frac{\tilde{D}v}{|\tilde{D}u|}, \nu \right\rangle \quad |\tilde{D}u| \text{-a.e. in } \mathbf{R}^n$$

$\frac{33}{34}$  and since both sides of (33) are zero  $|\tilde{D}u|$ -almost everywhere on  $|\langle \tilde{D}u, \nu \rangle|$ -  
 $\frac{35}{36}$  negligible sets, we conclude that

$$\frac{37}{38} \quad f\left(\tilde{u}(x) + h \left\langle \frac{\tilde{D}u}{|\tilde{D}u|}(x), \nu \right\rangle\right) - f(\tilde{u}(x))$$

$$\frac{39}{40} \quad \lim_{h \rightarrow 0} \frac{\quad}{h} = \left\langle \frac{\tilde{D}v}{|\tilde{D}u|}(x), \nu \right\rangle,$$

$\frac{41}{42}$

$\frac{1}{2}$   $\left| \widetilde{D}u \right|$ -a.e. in  $\mathbf{R}^n$ . Since  $\nu$  is arbitrary, by [Remarks 7.2 and 7.3](#) the restriction of  
 $\frac{2}{3}$   $f$  to the affine space  $T_x^u$  is differentiable at  $\tilde{u}(x)$  for  $\left| \widetilde{D}u \right|$ -almost every  $x \in \mathbf{R}^n$   
 $\frac{3}{4}$  and [\(26\)](#) holds.  $\square$

$\frac{5}{6}$  It follows from [\(13\)](#), [\(14\)](#), and [\(15\)](#) that

$$\frac{6}{7} \quad (36) \quad D(t_1, \dots, t_n) = \sum_{I \in \mathbf{n}} (-1)^{|I|-1} |I| \prod_{i \in I} t_i \prod_{j \in I} (D_j + \lambda_j t_j) \det \mathbf{A}^{(\lambda)}(\bar{I}|\bar{I}).$$

$\frac{8}{9}$  Let  $t_i = \hat{x}_i$ ,  $i = 1, \dots, n$ . Lemma 1 leads to

$$\frac{10}{11} \quad (37) \quad D(\hat{x}_1, \dots, \hat{x}_n) = \prod_{i \in \mathbf{n}} \hat{x}_i \sum_{I \in \mathbf{n}} (-1)^{|I|-1} |I| \text{per } \mathbf{A}^{(\lambda)}(I|I) \det \mathbf{A}^{(\lambda)}(\bar{I}|\bar{I}).$$

$\frac{12}{13}$  By [\(3\)](#), [\(13\)](#), and [\(37\)](#), we have the following result:

$\frac{14}{15}$  **THEOREM 7.2.**

$$\frac{15}{16} \quad (38) \quad H_c = \frac{1}{2n} \sum_{l=1}^n l (-1)^{l-1} A_l^{(\lambda)},$$

$\frac{17}{18}$  where

$$\frac{19}{20} \quad (39) \quad A_l^{(\lambda)} = \sum_{I_l \subseteq \mathbf{n}} \text{per } \mathbf{A}^{(\lambda)}(I_l|I_l) \det \mathbf{A}^{(\lambda)}(\bar{I}_l|\bar{I}_l), |I_l| = l.$$

$\frac{21}{22}$  It is worth noting that  $A_l^{(\lambda)}$  of [\(39\)](#) is similar to the coefficients  $b_l$  of the  
 $\frac{23}{24}$  characteristic polynomial of [\(10\)](#). It is well known in graph theory that the  
 $\frac{25}$  coefficients  $b_l$  can be expressed as a sum over certain subgraphs. It is interesting  
to see whether  $A_l$ ,  $\lambda = 0$ , structural properties of a graph.

$\frac{26}{27}$  We may call [\(38\)](#) a parametric representation of  $H_c$ . In computation, the  
 $\frac{28}{29}$  parameter  $\lambda_i$  plays very important roles. The choice of the parameter usually  
depends on the properties of the given graph. For a complete graph  $K_n$ , let  
 $\lambda_i = 1$ ,  $i = 1, \dots, n$ . It follows from [\(39\)](#) that

$$\frac{30}{31} \quad (40) \quad A_l^{(1)} = \begin{cases} n!, & \text{if } l = 1 \\ 0, & \text{otherwise.} \end{cases}$$

$\frac{32}{33}$  By [\(38\)](#)

$$\frac{34}{35} \quad (41) \quad H_c = \frac{1}{2}(n-1)!.$$

$\frac{36}{37}$  For a complete bipartite graph  $K_{n_1 n_2}$ , let  $\lambda_i = 0$ ,  $i = 1, \dots, n$ . By [\(39\)](#),

$$\frac{38}{39} \quad (42) \quad A_l = \begin{cases} -n_1! n_2! \delta_{n_1 n_2}, & \text{if } l = 2 \\ 0, & \text{otherwise.} \end{cases}$$

$\frac{40}{41}$  [Theorem 7.2](#) leads to

$$\frac{41}{42} \quad (43) \quad H_c = \frac{1}{n_1 + n_2} n_1! n_2! \delta_{n_1 n_2}.$$

Now, we consider an asymmetrical approach. [Theorem 3.3](#) leads to

$$(44) \quad \det \mathbf{K}(t = 1, t_1, \dots, t_n; l|l) \\ = \sum_{I \subseteq \mathbf{n} - \{l\}} (-1)^{|I|} \prod_{i \in I} t_i \prod_{j \in I} (D_j + \lambda_j t_j) \det \mathbf{A}^{(\lambda)}(\bar{I} \cup \{l\} | \bar{I} \cup \{l\}).$$

By (3) and (16) we have the following asymmetrical result:

**THEOREM 7.3.**

$$(45) \quad H_c = \frac{1}{2} \sum_{I \subseteq \mathbf{n} - \{l\}} (-1)^{|I|} \text{per } \mathbf{A}^{(\lambda)}(I|I) \det \mathbf{A}^{(\lambda)}(\bar{I} \cup \{l\} | \bar{I} \cup \{l\})$$

which reduces to Goulden–Jackson’s formula when  $\lambda_i = 0, i = 1, \dots, n$  [14].

## 8. Various font features of the `amsmath` package

8.1. *Bold versions of special symbols.* In the `amsmath` package `\boldsymbol` is used for getting individual bold math symbols and bold Greek letters—everything in math except for letters of the Latin alphabet, where you’d use `\mathbf`. For example,

```
A_\infty + \pi A_0 \sim
\mathbf{A}_{\boldsymbol{\infty}} \boldsymbol{+}
\boldsymbol{\pi} \mathbf{A}_{\boldsymbol{0}}
```

looks like this:

$$A_\infty + \pi A_0 \sim \mathbf{A}_\infty + \boldsymbol{\pi} \mathbf{A}_0$$

8.2. *“Poor man’s bold”.* If a bold version of a particular symbol doesn’t exist in the available fonts, then `\boldsymbol` can’t be used to make that symbol bold. At the present time, this means that `\boldsymbol` can’t be used with symbols from the `msam` and `msbm` fonts, among others. In some cases, poor man’s bold (`\pmb`) can be used instead of `\boldsymbol`:

```
\[\frac{\partial x}{\partial y} \bigg|_{\partial z}
\pmb{\bigg|_{\partial z}}
\frac{\partial y}{\partial z} \]
```

So-called “large operator” symbols such as  $\sum$  and  $\prod$  require an additional command, `\mathop`, to produce proper spacing and limits when `\pmb` is used. For further details see *The T<sub>E</sub>Xbook*.

$$\sum_{\substack{i < B \\ i \text{ odd}}} \prod_{\kappa} \kappa F(r_i) \quad \sum_{\substack{i < B \\ i \text{ odd}}} \prod_{\kappa} \kappa(r_i)$$

$\frac{1}{2}$  `\[\sum_{\substack{i<B\\\text{\$i\$ odd}}}`  
 $\frac{2}{2}$  `\prod_{\kappa} \kappa F(r_i)\qqquad`  
 $\frac{3}{2}$  `\mathop{\pmb{\sum}}_{\substack{i<B\\\text{\$i\$ odd}}}`  
 $\frac{4}{2}$  `\mathop{\pmb{\prod}}_{\kappa} \kappa(r_i)`  
 $\frac{5}{2}$  `\]`

 $\frac{6}{2}$  $\frac{7}{2}$ 

## 9. Compound symbols and other features

 $\frac{8}{2}$  $\frac{9}{2}$  $\frac{10}{2}$  $\frac{11}{2}$ 

9.1. *Multiple integral signs.* `\iint`, `\iiint`, and `\iiiiint` give multiple integral signs with the spacing between them nicely adjusted, in both text and display style. `\idotsint` gives two integral signs with dots between them.

 $\frac{12}{2}$  $\frac{13}{2}$  $\frac{14}{2}$  $\frac{15}{2}$  $\frac{16}{2}$  $\frac{17}{2}$  $\frac{18}{2}$  $\frac{19}{2}$  $\frac{20}{2}$ 

$$(46) \quad \iint_A f(x, y) dx dy \quad \iiint_A f(x, y, z) dx dy dz$$

$$(47) \quad \idotsint_A f(w, x, y, z) dw dx dy dz \quad \int \cdots \int_A f(x_1, \dots, x_k)$$

 $\frac{21}{2}$  $\frac{22}{2}$  $\frac{23}{2}$  $\frac{24}{2}$  $\frac{25}{2}$ 

9.2. *Over and under arrows.* Some extra over and under arrow operations are provided in the `amsmath` package. (Basic L<sup>A</sup>T<sub>E</sub>X provides `\overrightarrow` and `\overleftarrow`).

 $\frac{26}{2}$  $\frac{27}{2}$  $\frac{28}{2}$  $\frac{29}{2}$  $\frac{30}{2}$  $\frac{31}{2}$  $\frac{32}{2}$  $\frac{33}{2}$  $\frac{34}{2}$  $\frac{35}{2}$  $\frac{36}{2}$  $\frac{37}{2}$  $\frac{38}{2}$  $\frac{39}{2}$  $\frac{40}{2}$  $\frac{41}{2}$  $\frac{42}{2}$ 

$$\overrightarrow{\psi_\delta(t) E_t h} = \psi_\delta(t) E_t h$$

$$\overleftarrow{\psi_\delta(t) E_t h} = \psi_\delta(t) E_t h$$

$$\overleftrightarrow{\psi_\delta(t) E_t h} = \psi_\delta(t) E_t h$$

```

\begin{align*}
\overrightarrow{\psi_\delta(t) E_t h}& \\
=\underrightarrow{\psi_\delta(t) E_t h}& \\
\overleftarrow{\psi_\delta(t) E_t h}& \\
=\underleftarrow{\psi_\delta(t) E_t h}& \\
\overleftrightharrow{\psi_\delta(t) E_t h}& \\
=\underleftrightharrow{\psi_\delta(t) E_t h}& \\
\end{align*}

```

These all scale properly in subscript sizes:

 $\frac{36}{2}$  $\frac{37}{2}$  $\frac{38}{2}$  $\frac{39}{2}$  $\frac{40}{2}$  $\frac{41}{2}$  $\frac{42}{2}$ 

$$\int_{\overrightarrow{AB}} ax dx$$

$$\int_{\overleftarrow{AB}} ax dx$$

9.3. *Dots.* Normally you need only type `\dots` for ellipsis dots in a math formula. The main exception is when the dots fall at the end of the formula; then you need to specify one of `\dotsc` (series dots, after a comma), `\dotsb`

1 (binary dots, for binary relations or operators), `\dotsm` (multiplication dots),  
2 or `\dotsi` (dots after an integral). For example, the input

3 Then we have the series  $A_1, A_2, \dots$ ,  
4 the regional sum  $A_1 + A_2 + \dots$ ,  
5 the orthogonal product  $A_1 A_2 \dots$ ,  
6 and the infinite integral  
7  $\int \int \dots$ .  
8

9 produces

10 Then we have the series  $A_1, A_2, \dots$ , the regional sum  $A_1 + A_2 +$   
11  $\dots$ , the orthogonal product  $A_1 A_2 \dots$ , and the infinite integral

$$\int_{A_1} \int_{A_2} \dots$$

12 9.4. *Accents in math.* Double accents:  
13

14  $\hat{H} \check{C} \tilde{T} \acute{A} \grave{G} \dot{D} \ddot{D} \breve{B} \bar{B} \vec{V}$

15  $\hat{\hat{H}} \check{\check{C}} \tilde{\tilde{T}} \acute{\acute{A}} \grave{\grave{G}} \dot{\dot{D}} \ddot{\ddot{D}} \breve{\breve{B}} \bar{\bar{B}} \vec{\vec{V}}$   
16  $\hat{\hat{H}} \check{\check{C}} \tilde{\tilde{T}} \acute{\acute{A}} \grave{\grave{G}} \dot{\dot{D}} \ddot{\ddot{D}} \breve{\breve{B}} \bar{\bar{B}} \vec{\vec{V}}$   
17  $\hat{\hat{H}} \check{\check{C}} \tilde{\tilde{T}} \acute{\acute{A}} \grave{\grave{G}} \dot{\dot{D}} \ddot{\ddot{D}} \breve{\breve{B}} \bar{\bar{B}} \vec{\vec{V}}$   
18  $\hat{\hat{H}} \check{\check{C}} \tilde{\tilde{T}} \acute{\acute{A}} \grave{\grave{G}} \dot{\dot{D}} \ddot{\ddot{D}} \breve{\breve{B}} \bar{\bar{B}} \vec{\vec{V}}$   
19  $\hat{\hat{H}} \check{\check{C}} \tilde{\tilde{T}} \acute{\acute{A}} \grave{\grave{G}} \dot{\dot{D}} \ddot{\ddot{D}} \breve{\breve{B}} \bar{\bar{B}} \vec{\vec{V}}$   
20  $\hat{\hat{H}} \check{\check{C}} \tilde{\tilde{T}} \acute{\acute{A}} \grave{\grave{G}} \dot{\dot{D}} \ddot{\ddot{D}} \breve{\breve{B}} \bar{\bar{B}} \vec{\vec{V}}$   
21  $\hat{\hat{H}} \check{\check{C}} \tilde{\tilde{T}} \acute{\acute{A}} \grave{\grave{G}} \dot{\dot{D}} \ddot{\ddot{D}} \breve{\breve{B}} \bar{\bar{B}} \vec{\vec{V}}$   
22  $\hat{\hat{H}} \check{\check{C}} \tilde{\tilde{T}} \acute{\acute{A}} \grave{\grave{G}} \dot{\dot{D}} \ddot{\ddot{D}} \breve{\breve{B}} \bar{\bar{B}} \vec{\vec{V}}$

23 This double accent operation is complicated and tends to slow down the pro-  
24 cessing of a L<sup>A</sup>T<sub>E</sub>X file.

25 9.5. *Dot accents.* `\dddots` and `\ddddots` are available to produce triple and  
26 quadruple dot accents in addition to the `\dot` and `\ddot` accents already avail-  
27 able in L<sup>A</sup>T<sub>E</sub>X:

28  $\ddot{Q} \ddot{R}$   
29

30  $\dddots{Q} \ddddots{R}$

31 9.6. *Roots.* In the `amsmath` package `\leftroot` and `\uproot` allow you to  
32 adjust the position of the root index of a radical:

33  $\sqrt[\leftroot{-2}\uproot{2}\beta]{k}$

34 gives good positioning of the  $\beta$ :  
35

36  $\sqrt[\beta]{k}$

37 9.7. *Boxed formulas.* The command `\boxed` puts a box around its argu-  
38 ment, like `\fbox` except that the contents are in math mode:

39  $\boxed{W_t - F \subseteq V(P_i) \subseteq W_t}$

40  $\boxed{W_t - F \subseteq V(P_i) \subseteq W_t}$   
41  $\boxed{W_t - F \subseteq V(P_i) \subseteq W_t}$   
42

1 9.8. *Extensible arrows.* `\xleftarrow` and `\xrightarrow` produce arrows  
 2 that extend automatically to accommodate unusually wide subscripts or su-  
 3 perscripts. The text of the subscript or superscript are given as an optional  
 4 resp. mandatory argument: Example:

$$0 \xleftarrow[\zeta]{\alpha} F \times \triangle[n-1] \xrightarrow{\partial_0 \alpha(b)} E^{\partial_0 b}$$

5  
6  
7  
8 `\[0 \xleftarrow[\zeta]{\alpha} F \times \triangle[n-1]`  
 9 `\xrightarrow{\partial_0 \alpha(b)} E^{\partial_0 b}\]`

10 9.9. `\overset`, `\underset`, and `\sideset`. Examples:

$$\overset{*}{X} \quad X \quad \underset{*}{X}$$

11  
12  
13  
14 `\[\overset{*}{X} \qquad \underset{*}{X} \qquad \]`  
 15 `\overset{a}{\underset{b}{X}}\]`

16 The command `\sideset` is for a rather special purpose: putting symbols  
 17 at the subscript and superscript corners of a large operator symbol such as  $\sum$   
 18 or  $\prod$ , without affecting the placement of limits. Examples:

$$\overset{*}{\underset{*}{\prod}}_k \quad \sum'_{0 \leq i \leq m} E_i \beta x$$

19  
20  
21  
22 `\[\sideset{_*^*}_{_*^*} \prod_k \qquad`  
 23 `\sideset{}{'} \sum_{0 \leq i \leq m} E_i \beta x`  
 24 `\]`

25 9.10. *The `\text` command.* The main use of the command `\text` is for  
 26 words or phrases in a display:

$$\mathbf{y} = \mathbf{y}' \quad \text{if and only if} \quad y'_k = \delta_k y_{\tau(k)}$$

27  
28  
29  
30 `\[\mathbf{y} = \mathbf{y}' \quad \text{if and only if} \quad`  
 31 `y'_k = \delta_k y_{\tau(k)}\]`

32 9.11. *Operator names.* The more common math functions such as `\log`, `\sin`,  
 33 and `\lim` have predefined control sequences: `\log`, `\sin`, `\lim`. The `amsmath`  
 34 package provides `\DeclareMathOperator` and `\DeclareMathOperator*` for  
 35 producing new function names that will have the same typographical treat-  
 36 ment. Examples:

$$\|f\|_{\infty} = \operatorname{ess\,sup}_{x \in R^n} |f(x)|$$

37  
38  
39 `\[\norm{f}_{\infty} =`  
 40 `\operatorname{esssup}_{x \in R^n} \abs{f(x)}\]`  
 41  $\operatorname{meas}_1 \{u \in R_+^1 : f^*(u) > \alpha\} = \operatorname{meas}_n \{x \in R^n : |f(x)| \geq \alpha\} \quad \forall \alpha > 0.$   
 42

1  $\backslash[\backslash\text{meas}_1\{u\in R_{+^1}\text{colon } f^*(u)>\alpha\}$   
 2  $=\backslash\text{meas}_n\{x\in R^n\text{colon } \text{abs}\{f(x)\}\geq\alpha\}$   
 3  $\backslash\text{quad } \backslash\text{forall}\alpha>0.\backslash]$

4  $\backslash\text{esssup}$  and  $\backslash\text{meas}$  would be defined in the document preamble as

5  $\backslash\text{DeclareMathOperator}\{\backslash\text{esssup}\}\{\text{ess}\backslash,\text{sup}\}$   
 6  $\backslash\text{DeclareMathOperator}\{\backslash\text{meas}\}\{\text{meas}\}$

7  
 8 The following special operator names are predefined in the `amsmath` pack-  
 9 age:  $\backslash\text{varlimsup}$ ,  $\backslash\text{varliminf}$ ,  $\backslash\text{varinjlim}$ , and  $\backslash\text{varprojlim}$ . Here's what  
 10 they look like in use:

11  
 12 (48) 
$$\overline{\lim}_{n\rightarrow\infty} Q(u_n, u_n - u^\#) \leq 0$$

13  
 14 (49) 
$$\lim_{n\rightarrow\infty} |a_{n+1}| / |a_n| = 0$$

15  
 16 (50) 
$$\varinjlim (m_i^\lambda)^* \leq 0$$

17  
 18 (51) 
$$\varprojlim_{p\in S(A)} A_p \leq 0$$

19  $\backslash\text{begin}\{\text{align}\}$   
 20  $\&\backslash\text{varlimsup}_{n\rightarrow\infty}$   
 21  $\quad \backslash\text{mathcal}\{Q\}(u_n, u_n - u^\#)\leq 0\backslash\backslash$   
 22  $\&\backslash\text{varliminf}_{n\rightarrow\infty}$   
 23  $\quad \backslash\left\lvert\! \left\lvert a_{n+1}\right\rvert\right\rvert\backslash\right\rvert\backslash\left\lvert\! \left\lvert a_n\right\rvert\right\rvert=0\backslash\backslash$   
 24  $\&\backslash\text{varinjlim } (m_i^\lambda)^*\leq 0\backslash\backslash$   
 25  $\&\backslash\text{varprojlim}_{p\in S(A)} A_p\leq 0$   
 26  $\backslash\text{end}\{\text{align}\}$

27 9.12.  $\backslash\text{mod}$  and its relatives. The commands  $\backslash\text{mod}$  and  $\backslash\text{pod}$  are variants  
 28 of  $\backslash\text{pmod}$  preferred by some authors;  $\backslash\text{mod}$  omits the parentheses, whereas  $\backslash\text{pod}$   
 29 omits the 'mod' and retains the parentheses. Examples:

30  
 31 (52) 
$$x \equiv y + 1 \pmod{m^2}$$

32  
 33 (53) 
$$x \equiv y + 1 \pmod{m^2}$$

34  
 35 (54) 
$$x \equiv y + 1 \pmod{m^2}$$

36  $\backslash\text{begin}\{\text{align}\}$   
 37  $x\equiv y+1\pmod{m^2}\backslash\backslash$   
 38  $x\equiv y+1\pmod{m^2}\backslash\backslash$   
 39  $x\equiv y+1\pmod{m^2}$   
 40  $\backslash\text{end}\{\text{align}\}$

41 9.13. *Fractions and related constructions.* The usual notation for binomi-  
 42 als is similar to the fraction concept, so it has a similar command  $\backslash\text{binom}$  with

1 two arguments. Example:

$$\begin{aligned}
 & \sum_{\gamma \in \Gamma_C} I_\gamma = 2^k - \binom{k}{1} 2^{k-1} + \binom{k}{2} 2^{k-2} \\
 & + \dots + (-1)^l \binom{k}{l} 2^{k-l} + \dots + (-1)^k \\
 & = (2-1)^k = 1
 \end{aligned}$$

9 `\begin{equation}`

10 `\begin{split}`

11 `[\sum_{\gamma \in \Gamma_C} I_\gamma &`

12 `=2^k - \binom{k}{1} 2^{k-1} + \binom{k}{2} 2^{k-2} \\\`

13 `&\quad + \dots + (-1)^l \binom{k}{l} 2^{k-l} \\\`

14 `+ \dots + (-1)^k \\\`

15 `&= (2-1)^k = 1`

16 `\end{split}`

17 `\end{equation}`

18 There are also abbreviations

19 `\dfrac`      `\dbinom`

20 `\tfrac`      `\tbinom`

21 for the commonly needed constructions

22 `{\displaystyle\frac ... }    {\displaystyle\binom ... }`

23 `{\textstyle\frac ... }        {\textstyle\binom ... }`

24 The generalized fraction command `\genfrac` provides full access to the  
25 six T<sub>E</sub>X fraction primitives:

$$\text{(56)} \quad \text{\over:} \frac{n+1}{2} \quad \text{\overwithdelims:} \left\langle \frac{n+1}{2} \right\rangle$$

$$\text{(57)} \quad \text{\atop:} \frac{n+1}{2} \quad \text{\atopwithdelims:} \left( \frac{n+1}{2} \right)$$

$$\text{(58)} \quad \text{\above:} \frac{n+1}{2} \quad \text{\abovewithdelims:} \left[ \frac{n+1}{2} \right]$$

34 `\text{\cn{over}: }&\genfrac{}{}{}{}{n+1}{2}&`

35 `\text{\cn{overwithdelims}: }&`

36 `\genfrac{\langle\rangle}{\rangle}{}{}{n+1}{2} \\\`

37 `\text{\cn{atop}: }&\genfrac{}{}{0pt}{}{n+1}{2}&`

38 `\text{\cn{atopwithdelims}: }&`

39 `\genfrac{()}{)}{0pt}{}{n+1}{2} \\\`

40 `\text{\cn{above}: }&\genfrac{}{}{1pt}{}{n+1}{2}&`

41 `\text{\cn{abovewithdelims}: }&`

42



`\genfrac{[]{}{1pt}{}{n+1}{2}`

9.14. *Continued fractions.* The continued fraction

$$(59) \quad \frac{1}{\sqrt{2} + \frac{1}{\sqrt{2} + \frac{1}{\sqrt{2} + \frac{1}{\sqrt{2} + \dots}}}}$$

can be obtained by typing

```
\frac{1}{\sqrt{2}+
\frac{1}{\sqrt{2}+
\frac{1}{\sqrt{2}+
\frac{1}{\sqrt{2}+
\frac{1}{\sqrt{2}+\dotsb
}}}}}
```

Left or right placement of any of the numerators is accomplished by using `\frac[1]` or `\frac[r]` instead of `\frac`.

9.15. *Smash.* In `amsmath` there are optional arguments `t` and `b` for the plain TeX command `\smash`, because sometimes it is advantageous to be able to ‘smash’ only the top or only the bottom of something while retaining the natural depth or height. In the formula  $X_j = (1/\sqrt{\lambda_j})X'_j$  `\smash[b]` has been used to limit the size of the radical symbol.

```
\$X_j=(1/\sqrt{\smash[b]{\lambda_j}})X'_j'
```

Without the use of `\smash[b]` the formula would have appeared thus:  $X_j = (1/\sqrt{\lambda_j})X'_j$ , with the radical extending to encompass the depth of the subscript  $j$ .

9.16. *The ‘cases’ environment.* ‘Cases’ constructions like the following can be produced using the `cases` environment.

$$(60) \quad P_{r-j} = \begin{cases} 0 & \text{if } r-j \text{ is odd,} \\ r!(-1)^{(r-j)/2} & \text{if } r-j \text{ is even.} \end{cases}$$

```
\begin{equation} P_{r-j}=
\begin{cases}
0& \text{\text{if } $r-j$ is odd},\
r!\,(-1)^{(r-j)/2}& \text{\text{if } $r-j$ is even}.
\end{cases}
\end{equation}
```

1 Notice the use of `\text` and the embedded math.

2           9.17. *Matrix*. Here are samples of the matrix environments, `\matrix`,  
3 `\pmatrix`, `\bmatrix`, `\Bmatrix`, `\vmatrix` and `\Vmatrix`:  
4

5

6

7 (61)      $\vartheta$     $\varrho$      $\begin{pmatrix} \vartheta & \varrho \\ \varphi & \varpi \end{pmatrix}$     $\begin{bmatrix} \vartheta & \varrho \\ \varphi & \varpi \end{bmatrix}$     $\begin{Bmatrix} \vartheta & \varrho \\ \varphi & \varpi \end{Bmatrix}$     $\begin{vmatrix} \vartheta & \varrho \\ \varphi & \varpi \end{vmatrix}$     $\begin{Vmatrix} \vartheta & \varrho \\ \varphi & \varpi \end{Vmatrix}$

8

9

10

11 `\begin{matrix}`  
12 `\vartheta& \varrho\\ \varphi& \varpi`  
13 `\end{matrix}\quad`  
14 `\begin{pmatrix}`  
15 `\vartheta& \varrho\\ \varphi& \varpi`  
16 `\end{pmatrix}\quad`  
17 `\begin{bmatrix}`  
18 `\vartheta& \varrho\\ \varphi& \varpi`  
19 `\end{bmatrix}\quad`  
20 `\begin{Bmatrix}`  
21 `\vartheta& \varrho\\ \varphi& \varpi`  
22 `\end{Bmatrix}\quad`  
23 `\begin{vmatrix}`  
24 `\vartheta& \varrho\\ \varphi& \varpi`  
25 `\end{vmatrix}\quad`  
26 `\begin{Vmatrix}`  
27 `\vartheta& \varrho\\ \varphi& \varpi`  
28 `\end{Vmatrix}`

29

30

31           To produce a small matrix suitable for use in text, use the `smallmatrix`  
32 environment.

33

34 `\begin{math}`  
35     `\bigl( \begin{smallmatrix}`  
36         `a&b\\ c&d`  
37     `\end{smallmatrix} \bigl)`  
38 `\end{math}`

39

40 To show the effect of the matrix on the surrounding lines of a paragraph, we  
41 put it here:  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  and follow it with enough text to ensure that there will be  
42 at least one full line below the matrix.

`\hdotsfor{number}` produces a row of dots in a matrix spanning the given number of columns:

$$W(\Phi) = \begin{vmatrix} \frac{\varphi}{(\varphi_1, \varepsilon_1)} & 0 & \dots & 0 \\ \frac{\varphi^{k_{n2}}}{(\varphi_2, \varepsilon_1)} & \frac{\varphi}{(\varphi_2, \varepsilon_2)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \frac{\varphi^{k_{n1}}}{(\varphi_n, \varepsilon_1)} & \frac{\varphi^{k_{n2}}}{(\varphi_n, \varepsilon_2)} & \dots & \frac{\varphi^{k_{nn-1}}}{(\varphi_n, \varepsilon_{n-1})} & \frac{\varphi}{(\varphi_n, \varepsilon_n)} \end{vmatrix}$$

```

10 \[W(\Phi)= \begin{Vmatrix}
11 \dfrac{\varphi}{(\varphi_1, \varepsilon_1)}&0&\dots&0\\
12 \dfrac{\varphi^{k_{n2}}}{(\varphi_2, \varepsilon_1)}&\frac{\varphi}{(\varphi_2, \varepsilon_2)}&\dots&0\\
13 \dfrac{\varphi}{(\varphi_2, \varepsilon_1)} & \frac{\varphi}{(\varphi_2, \varepsilon_2)} & \dots & 0 \\
14 \hdotsfor{5}\\
15 \dfrac{\varphi^{k_{n1}}}{(\varphi_n, \varepsilon_1)}&\frac{\varphi^{k_{n2}}}{(\varphi_n, \varepsilon_2)}&\dots&\frac{\varphi^{k_{nn-1}}}{(\varphi_n, \varepsilon_{n-1})}&\frac{\varphi}{(\varphi_n, \varepsilon_n)} \\
16 \dfrac{\varphi^{k_{n2}}}{(\varphi_2, \varepsilon_1)} & \frac{\varphi}{(\varphi_2, \varepsilon_2)} & \dots & 0 \\
17 \dfrac{\varphi^{k_{n1}}}{(\varphi_n, \varepsilon_1)} & \frac{\varphi^{k_{n2}}}{(\varphi_n, \varepsilon_2)} & \dots & \frac{\varphi^{k_{nn-1}}}{(\varphi_n, \varepsilon_{n-1})} & \frac{\varphi}{(\varphi_n, \varepsilon_n)} \\
18 \dfrac{\varphi}{(\varphi_2, \varepsilon_1)} & \frac{\varphi}{(\varphi_2, \varepsilon_2)} & \dots & 0 \\
19 \end{Vmatrix}\]
```

The spacing of the dots can be varied through use of a square-bracket option, for example, `\hdotsfor[1.5]{3}`. The number in square brackets will be used as a multiplier; the normal value is 1.

9.18. *The `\substack` command.* The `\substack` command can be used to produce a multiline subscript or superscript: for example

```

25 \sum_{\substack{0 \leq i \leq m \\ 0 < j < n}} P(i, j)
```

produces a two-line subscript underneath the sum:

$$(62) \quad \sum_{\substack{0 \leq i \leq m \\ 0 < j < n}} P(i, j)$$

A slightly more generalized form is the `subarray` environment which allows you to specify that each line should be left-aligned instead of centered, as here:

Maybe "... as below"?

$$(63) \quad \sum_{\subarray{0 \leq i \leq m \\ 0 < j < n}} P(i, j)$$

```

38 \sum_{\begin{subarray}{l}
39 0 \leq i \leq m \\
40 \end{subarray}}
41 P(i, j)
```

1 9.19. *Big-g-g delimiters.* Here are some big delimiters, first in `\normalsize`:

2

$$\left( \mathbf{E}_y \int_0^{t_\varepsilon} L_{x,y^x(s)} \varphi(x) ds \right)$$

3

4

```
5 \[\biggl(\mathbf{E}_{\mathbf{y}}
6 \int_0^{\mathbf{t}_{\mathbf{varepsilon}}}\mathbf{L}_{\mathbf{x},\mathbf{y}^{\mathbf{x}}(s)}\mathbf{\varphi}(\mathbf{x})\,d\mathbf{s}
7 \biggr)
8 \]
```

9 and now in `\Large` size:

10

$$\left( \mathbf{E}_y \int_0^{t_\varepsilon} L_{x,y^x(s)} \varphi(x) ds \right)$$

11

12

```
13 {\Large
14 \[\biggl(\mathbf{E}_{\mathbf{y}}
15 \int_0^{\mathbf{t}_{\mathbf{varepsilon}}}\mathbf{L}_{\mathbf{x},\mathbf{y}^{\mathbf{x}}(s)}\mathbf{\varphi}(\mathbf{x})\,d\mathbf{s}
16 \biggr)
17 \biggr)
18 \]}
```

19 9.20. *Acknowledgements.* The authors are grateful to NASA (grant 123456)

20 and NIH. They acknowledge the generous help of other agencies .

21

## 22 **References**

- 23
- 24 [1] V. I. ARNOLD, *Mathematical Methods of Classical Mechanics*, second ed., *Grad-*
- 25 *uate Texts in Mathematics* **60**, Springer, New York, 1989.
- 26 [2] W. DIFFIE and E. HELLMAN, New directions in cryptography, *IEEE Transac-*
- 27 *tions on Information Theory* **22** no. 5 (1976), 644–654.
- 28 [3] D. H. FREMLIN, Cichon’s diagram, presented at the Séminaire Initiation à
- 29 l’Analyse, G. Choquet, M. Rogalski, J. Saint Raymond, at the Université Pierre
- 30 et Marie Curie, Paris, 23e année., 1983/1984.
- 31 [4] D. H. FREMLIN, *Topological Riesz Spaces and Measure Theory*, Cambridge Uni-
- 32 *versity Press*, 2008.
- 33 [5] I. P. GOULDEN and D. M. JACKSON, The enumeration of directed closed Eu-
- 34 *ler trails and directed Hamiltonian circuits by Langrangian methods*, *European*
- 35 *Journal of Combinatorics* **2** (1981), 131–212.
- 36 [6] C. DE GROOT, D. WÜRTZ, M. HANF, R. PEIKERT, T. KOLLER, and K. H.
- 37 *HOFFMANN*, Stochastic optimization—efficient algorithms to solve complex prob-
- 38 *lems*, in *System Modelling and Optimization, Proceedings of the Fifteenth IFIP*
- 39 *Conference (Zürich)* (P. KALL, ed.), Springer-Verlag, 1992, pp. 546–555.
- 40 [7] F. HARARY and E. M. PALMER, *Graphical Enumeration*, Academic Press, 1973.
- 41 [8] R. IMPAGLIAZZO, L. LEVIN, and M. LUBY, Pseudo-random generation from one-
- 42 *way functions*, in *Proc. 21st STOC (Seattle, WA, USA)*, ACM, New York, 1989,
- pp. 12–24.

- 1 [9] D. E. KNUTH, *The T<sub>E</sub>Xbook*, with illustrations by Duane Bibby, *Computers &*  
2 *Typesetting A*, Addison-Wesley Publishing Company, Reading, MA, 1994.
- 3 [10] M. KOJIMA, S. MIZUNO, and A. YOSHISE, *A New Continuation Method for Com-*  
4 *plementarity Problems With Uniform p-Functions*, Tech. Report B-194, Tokyo  
5 Inst. of Technology, Dept. of Information Sciences, Tokyo, 1987.
- 6 [11] M. KOJIMA, S. MIZUNO, and A. YOSHISE, *A Polynomial-Time Algorithm For a*  
7 *Class of Linear Complementarity Problems*, Tech. Report B-193, Tokyo Inst. of  
8 Technology, Dept. of Information Sciences, Tokyo, 1987.
- 9 [12] H. W. LENSTRA, JR. and F. OORT, Simple abelian varieties having a prescribed  
10 formal isogeny type., *J. Pure Appl. Algebra* **4** (1974), 47–53. [MR 0279.14009](#).  
11 [Zbl 50:7163](#). [https://doi.org/10.1016/0022-4049\(74\)90029-2](https://doi.org/10.1016/0022-4049(74)90029-2).
- 12 [13] C. J. LIU and Y. CHOW, On operator and formal sum methods for graph enu-  
13 meration problems, *SIAM Journal of Algorithms and Discrete Methods* **5** (1984),  
14 384–438.
- 15 [14] M. MARCUS and H. MINC, A survey of matrix theory and matrix inequalities,  
16 *Complementary Series in Mathematics* **14** (1964), 21–48.
- 17 [15] A. D. MICHAL, Differential calculus in linear topological spaces, *Proc. nat. Acad.*  
18 *Sci. USA* **24** (1938), 340–342. [JFM 64.0366.02](#).
- 19 [16] A. D. MICHAL, *Matrix and Tensor Calculus*, *GALCIT Aeronautical Series*, John  
20 Wiley & Sons, Inc.; Chapman & Hall, Ltd., New York; London, 1948.
- 21 [17] A. MINASYAN and D. OSIN, *Normal Automorphisms of Relatively Hyperbolic*  
22 *Groups*, 2008. [arXiv 0809.2408](#).
- 23 [18] S. MIZUNO, A. YOSHISE, and T. KIKUCHI, *Practical Polynomial Time Algo-*  
24 *rithms for Linear Complementarity Problems*, Tech. Report 13, Tokyo Inst. of  
25 Technology, Dept. of Industrial Engineering and Management, Tokyo, April 1988.
- 26 [19] R. D. MONTEIRO and I. ADLER, *Interior Path Following Primal-Dual Algo-*  
27 *rithms, Part II: Quadratic Programming*, Working paper, Dept. of Industrial En-  
28 gineering and Operations Research, August 1987.
- 29 [20] E. M. STEIN, *Singular Integrals and Differentiability Properties of Functions*,  
30 Princeton Univ. Press, Princeton, NJ, 1970.
- 31 [21] Y. YE, *Interior Algorithms for Linear, Quadratic and Linearly Constrained Con-*  
32 *convex Programming*, Ph.D. thesis, Stanford Univ., Dept. of Engineering–Economic  
33 Systems, Palo Alto, CA, July 1987.
- 34 [22] YU. G. ZARHIN, Abelian varieties having a reduction of K3 type, *Duke Math J.*  
35 **65** no. 3 (1992), 17 pp. [MR 1154181](#). [Zbl 0774.14039](#).
- 36 [23] YU. G. ZARHIN, *Algebra and Cryptography*, Private Communication.
- 37 [24] YU. G. ZARHIN, *On Abel Groups*, Private Communication.

(Received: December 24, 2004)

(Revised: April 12, 2006)

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