

Curves

(PARI-GP version 2.19.0)

Conics

find a rational point on a conic, ${}^t x G x = 0$ `qfsolve(G)`
[*H*, *U*] such that $H = c^t U G U$ has minimat `qfminimize(G)`
quadratic Hilbert symbol (at *p*) `hilbert(x, y, {p})`
all solutions in $\mathbf{Q}^3$ of ternary form `qfparam(G, x)`

Elliptic curves

An elliptic curve is initially given by 5-tuple $v = [a_1, a_2, a_3, a_4, a_6]$ attached to Weierstrass model or simply $[a_4, a_6]$. It must be converted to an *ell* struct.

Initialize *ell* struct over domain *D* `E = ellinit(v, {D = 1})`
over $\mathbf{Q}$ `D = 1`
over $\mathbf{F}_p$ `D = p`
over $\mathbf{F}_q$, $q = p^f$ `D = ffgen([p, f])`
over $\mathbf{Q}_p$, precision *n* `D = O(pn)`
over $\mathbf{C}$, current bitprecision `D = 1.0`
over number field *K* `D = nf`

Points are [*x*,*y*], the origin is [0]. Struct members:

- All domains: **E.a1,a2,a3,a4,a6**, **b2,b4,b6,b8**, **c4,c6**, **disc**, **j**
- *E* defined over $\mathbf{R}$ or $\mathbf{C}$
x-coords. of points of order 2 **E.roots**
periods / quasi-periods **E.omega**, **E.eta**
volume of complex lattice **E.area**
- *E* defined over $\mathbf{Q}_p$
residual characteristic **E.p**
If $|j|_p > 1$: Tate's $[u^2, u, q, [a, b], \mathcal{L}]$ **E.tate**
- *E* defined over $\mathbf{F}_q$
characteristic **E.p**
 $\#E(\mathbf{F}_q)$ /cyclic structure/generators **E.no**, **E.cyc**, **E.gen**
- *E* defined over $\mathbf{Q}$
generators of $E(\mathbf{Q})$ (require **elldata**) **E.gen**
 $[a_1, a_2, a_3, a_4, a_6]$ from *j*-invariant **ellfromj(*j*)**
cubic/quartic/biquadratic to Weierstrass **ellfromeqn(*eq*)**
add points $P + Q$ / $P - Q$ **elladd(*E*, *P*, *Q*)**, **ellsub**
negate point **ellneg(*E*, *P*)**
compute $n \cdot P$ **ellmul(*E*, *P*, *n*)**
sum of Galois conjugates of *P* **elltrace(*E*, *P*)**
check if *P* is on *E* **ellisoncurve(*E*, *P*)**
order of torsion point *P* **ellorder(*E*, *P*)**
y-coordinates of point(*s*) for *x* **ellordinate(*E*, *x*)**
 $[\wp(z), \wp'(z)] \in E(\mathbf{C})$ attached to $z \in \mathbf{C}$ **ellztopoint(*E*, *z*)**
 $z \in \mathbf{C}$ such that $P = [\wp(z), \wp'(z)]$ **ellpointtoz(*E*, *P*)**
 $z \in \tilde{\mathbf{Q}}^*/q^{\mathbf{Z}}$ to $P \in E(\tilde{\mathbf{Q}}_p)$ **ellztopoint(*E*, *z*)**
 $P \in E(\tilde{\mathbf{Q}}_p)$ to $z \in \tilde{\mathbf{Q}}^*/q^{\mathbf{Z}}$ **ellpointtoz(*E*, *P*)**
- Change of Weierstrass models**, using $v = [u, r, s, t]$
compose/invert changes **ellchangecompose**, **ellchangeinvert**
change curve *E* using *v* **ellchangecurve(*E*, *v*)**
change point *P* using *v* **ellchangepoint(*P*, *v*)**
change point *P* using inverse of *v* **ellchangepointinv(*P*, *v*)**
is *E* isomorphic to *F*? **ellsisom(*E*, *F*)**

Twists and isogenies

quadratic twist **elltwtst(*E*, *d*)**
n-division polynomial $f_n(x)$ **elldivpol(*E*, *n*, {*x*})**
 $[n]P = (\phi_n \psi_n; \omega_n; \psi_n^3)$; return (ϕ_n, ψ_n^2) **ellxn(*E*, *n*, {*x*})**
isogeny from *E* to *E*/*G* **ellisogeny(*E*, *G*)**
apply isogeny to *g* (point or isogeny) **ellisogenyapply(*f*, *g*)**
torsion subgroup with generators **elltors(*E*)**

Formal group

formal exponential, *n* terms **ellformalexp(*E*, {*n*}, {*x*})**
formal logarithm, *n* terms **ellformalog(*E*, {*n*}, {*x*})**
 $\log_E(-x(P)/y(P)) \in \mathbf{Q}_p$; $P \in E(\mathbf{Q}_p)$ **ellpadiclog(*E*, *p*, *n*, *P*)**
P in the formal group **ellformalpoint(*E*, {*n*}, {*x*})**
 $[\omega/dt, x\omega/dt]$ **ellformaldifferential(*E*, {*n*}, {*x*})**
 $w = -1/y$ in parameter $-x/y$ **ellformalw(*E*, {*n*}, {*x*})**

Elliptic curves over finite fields, Pairings

random point on *E* **random(*E*)**
 $\#E(\mathbf{F}_q)$ **ellcard(*E*)**
 $\#E(\mathbf{F}_q)$ with almost prime order **ellsea(*E*, {*tors*})**
structure $\mathbf{Z}/d_1\mathbf{Z} \times \mathbf{Z}/d_2\mathbf{Z}$ of $E(\mathbf{F}_q)$ **ellgroup(*E*)**
is *E* supersingular? **ellissupersingular(*E*)**
random supersingular *j*-invariant over F_p^2 **ellsupersingularj(*p*)**
Weil pairing of *m*-torsion pts *P*, *Q* **ellweilpairing(*E*, *P*, *Q*, *m*)**
Tate pairing of *P*, *Q*; *P* *m*-torsion **elltatepairing(*E*, *P*, *Q*, *m*)**
Discrete log, find *n* s.t. $P = [n]Q$ **elllog(*E*, *P*, *Q*, {*ord*})**

Elliptic curves over Q

Reduction, minimal model

minimal model of *E*/ $\mathbf{Q}$ **ellminimalmodel(*E*, {*fv*})**
quadratic twist of minimal conductor **ellminimaltwist(*E*)**
 $[k]P$ with good reduction **ellnonsingularmultiple(*E*, *P*)**
E supersingular at *p*? **ellissupersingular(*E*, *p*)**
affine points of naïve height $\leq h$ **ellratpoints(*E*, *h*)**

Complex heights

canonical height of *P* **ellheight(*E*, *P*)**
canonical bilinear form taken at *P*, *Q* **ellheight(*E*, *P*, *Q*)**
height regulator matrix for pts in *L* **ellheightmatrix(*E*, *L*)**

p-adic heights

cyclotomic *p*-adic height of $P \in E(\mathbf{Q})$ **ellpadicheight(*E*, *p*, *n*, *P*)**
... bilinear form at $P, Q \in E(\mathbf{Q})$ **ellpadicheight(*E*, *p*, *n*, *P*, *Q*)**
... matrix at vector for pts in *L* **ellpadicheightmatrix(*E*, *p*, *n*, *L*)**
... regulator for canonical height **ellpadicregulator(*E*, *p*, *n*, *Q*)**
Frobenius on $\mathbf{Q}_p \otimes H_{dR}^1(E/\mathbf{Q})$ **ellpadicfrobenius(*E*, *p*, *n*)**
slope of unit eigenvector of Frobenius **ellpadics2(*E*, *p*, *n*)**

Isogenous curves

matrix of isogeny degrees for $\mathbf{Q}$ -isog. curves **ellisomat(*E*)**
tree of prime degree isogenies **ellisotree(*E*)**
a modular equation of prime degree *N* **ellmodulareqn(*N*)**

L-function

char. poly. of Frobenius **ellcharpoly(*E*, *p*)**
p-th coeff *a_p* of *L*-function, *p* prime **ellap(*E*, *p*)**
k-th coeff *a_k* of *L*-function **ellak(*E*, *k*)**
L(*E*, *s*) (using less memory than **lfun**) **elllseries(*E*, *s*)**
 $L^{(r)}(E, 1)$ (using less memory than **lfun**) **ellL1(*E*, *r*)**
order of vanishing at 1 **ellanalyticrank(*E*, {*eps*})**
root number for $L(E, \cdot)$ at *p* **ellrootno(*E*, {*p*})**
modular parametrization of *E* **elltaniyama(*E*)**
degree of modular parametrization **ellmoddegree(*E*)**
compare with $H^1(X_0(N), \mathbf{Z})$ (for $E' \rightarrow E$) **ellweilcurve(*E*)**
Manin constant of *E* **ellmaninconstant(*E*)**

p-adic *L* function $L_p^{(r)}(E, d, \chi^s)$ **ellpadicL(*E*, *p*, *n*, {*s*}, {*r*}, {*d*})**
BSD conjecture for $L_p^{(r)}(E_D, \chi^0)$ **ellpadicbsd(*E*, *p*, *n*, {*D* = 1})**
Iwasawa invariants for $L_p(E_D, \tau^i)$ **ellpadiclambdamu(*E*, *p*, *D*, *i*)**

Rational points

a Heegner point on *E* of rank 1 **ellheegner(*E*)**
... on the twist E_D (of rank 1) **ellheegnertwist(*E*, *D*)**
attempt to compute $E(\mathbf{Q})$ **ellrank(*E*, {*effort*}, {*points*})**
initialize for later **ellrank** calls, **ellrankinit(*E*)**
saturate $(P_1, \dots, P_n)$ wrt. primes $\leq B$ **ellsaturation(*E*, *P*, *B*)**
2-covers of the curve *E* **ell2cover(*E*)**

Elldata package, Cremona's database:

db code "11a1" $\leftrightarrow$ [*conductor*, *class*, *index*] **ellconvertname(*s*)**
generators of Mordell-Weil group **ellgenerators(*E*)**
look up *E* in database **ellidentify(*E*)**
all curves matching criterion **ellsearch(*N*)**
loop over curves with cond. from *a* to *b* **forell(*E*, *a*, *b*, *seq*)**

Elliptic curves over number field *K*

coeff *a_p* of *L*-function **ellap(*E*, *p*)**
Kodaira type of $\mathfrak{p}$ -fiber of *E* **elllocalred(*E*, *p*)**
integral model of *E*/*K* **ellintegralmodel(*E*, {*fv*})**
minimal model of *E*/*K* **ellminimalmodel(*E*, {*fv*})**
minimal discriminant of *E*/*K* **ellminimaldisc(*E*)**
cond, min mod, Tamagawa num [*N*, *v*, *c*] **ellglobalred(*E*)**
global Tamagawa number **elltamagawa(*E*)**
test if *E* has CM **elliscm(*E*)**
 $P \in E(K)$ *n*-divisible? $[n]Q = P$ **ellisdivisible(*E*, *P*, *n*, {*Q*})**

L-function

A domain $D = [c, w, h]$ in initialization mean we restrict $s \in \mathbf{C}$ to domain $|\Re(s) - c| < w$, $|\Im(s)| < h$; $D = [w, h]$ encodes $[1/2, w, h]$ and $[h]$ encodes $D = [1/2, 0, h]$ (critical line up to height *h*).
vector of first *n* *a_k*'s in *L*-function **ellan(*E*, *n*)**
init $L^{(k)}(E, s)$ for $k \leq n$ **L = lfunitit(*E*, *D*, {*n* = 0})**
compute $L(E, s)$ (*n*-th derivative) **lfun(*L*, *s*, {*n* = 0})**
 $L(E, 1, r)/(r! \cdot R \cdot \#Sha)$ assuming BSD **ellbsd(*E*)**

Based on an earlier version by Joseph H. Silverman

July 2026 v2.40. Copyright © 2026 K. Belabas

Permission is granted to make and distribute copies of this card provided the copyright and this permission notice are preserved on all copies.

Send comments and corrections to (Karim.Belabas@math.u-bordeaux.fr)

Hyperelliptic curves

A hyperelliptic curve C is given by a pair $[P, Q]$ ($y^2 + Qy = P$ with $Q^2 + 4P$ squarefree) or a single squarefree polynomial P ($y^2 = P$).
check if $[x, y]$ is on C `hyperellisoncurve(C, [x, y])`
 y -coordinates of point(s) for x `hyperellordinate(C, x)`
Igusa invariants for C of genus 2 `genus2igusa(C)`
period matrix `hyperellperiods(C)`
affine rational points of height $\leq h$ `hyperellratpoints(C, h)`
Change of Weierstrass models, using $m = [e, [a, b, c, d], H]$
compose changes `hyperellchangecompose(C, m1, m2)`
invert change `hyperellchangeinvert(C, m)`
change curve C using m `hyperellchangecurve(C, m)`
change point P using m `hyperellchangepoint(P, m)`
change point P using inverse of m `hyperellchangepointinv(P, v)`
automorphisms of C `hyperellauto(C, {nf})`
is C_1 isomorphic to C_2 ? `hyperellisom(C1, C2)`
Cremona-Stoll reduction `hyperellred(C)`
Reduction, minimal model
discriminant of C `hyperelldisc(C)`
minimal discriminant of integral C `hyperellminimaldisc(C)`
minimal model of integral C `hyperellminimalmodel(C)`
maximal chain of minimal models at `hyperellxtremalmodels(C, p)`
reduction of $y^2 + Qy = P$ (genus 2) `genus2red(C, {p})`
Frobenius
char. poly. of Frobenius `genus2charpoly(C, p)`
matrix of Frobenius on $\mathbf{Q}_p \otimes H^1_{dR}$ `hyperellpadicfrobenius`
 $P, Q \in \mathbf{F}_q[X]$; char. poly. of Frobenius `hyperellcharpoly(C)`

Curves

(PARI-GP version 2.19.0)

Elliptic & Modular Functions

$w = [\omega_1, \omega_2]$ or ell struct (**E.omega**), $\tau = \omega_1/\omega_2$.
arithmetic-geometric mean `agm(x, y)`
elliptic j -function $1/q + 744 + \dots$ `ellj(x)`
Weierstrass $\sigma/\wp/\zeta$ function `ellsigma(w, z), ellwp, ellzeta`
periods/quasi-periods `ellperiods(E, {flag}), elleta(w)`
 $(\omega_1, \omega_2), (g_2, g_3), (e_1, e_2, e_3), (\eta_1, \eta_2)$ `ellweierstrass(w)`
 $(2i\pi/\omega_2)^k E_k(\tau)$ `elleisnum(w, k)`
modified Dedekind η func. $\prod_n (1 - q^n)$ `eta(x)`
 $\dots$ true η function $q^{1/24} \prod_n (1 - q^n)$ `eta(x, 1)`
Dedekind sum $s(h, k)$ `sumdedekind(h, k)`
Riemann-Jacobi theta functions $\vartheta_{a,b}(z; \tau)$ `theta(z, t, [a, b])`
 $\dots$ Thetanullwerte (at $z = 0$) `thetanull(t, {[a, b]})`
Jacobi sine-theta function $\theta(q; z)$ `theta(q, z)`
 $\dots \theta^{(k)}(q, 0)$ `thetanullk(q, k)`
Jacobi elliptic functions (sn, cn, dn) `elljacobi(z, k)`
Weber's f functions `weber(x, {flag})`
modular pol. of level N `polmodular(N, {inv = j})`
Hilbert class polynomial for $\mathbf{Q}(\sqrt{D})$ `polclass(D, {inv = j})`